ISRAEL JOURNAL OF MATHEMATICS 120 (2000), 361–427

ADJOINT SELMER GROUPS AS IWASAWA MODULES

BY

HARUZO HIDA*

Department of Mathematics, UCLA
Los Angeles, CA 90095-1555, USA
e-mail: hida@math.ucla.edu

To the memory of Professor Kenkichi Iwasawa

ABSTRACT

We fix a prime p. In this paper, starting from a given Galois representation φ having values in p-adic points of a classical group G, we study the adjoint action of φ on the p-adic Lie algebra of the derived group of G. We call this new Galois representation the adjoint representation $\mathrm{Ad}(\varphi)$ of φ. Under a suitable p-ordinarity condition (and ramification conditions outside p), we define, following Greenberg, the Selmer group $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/L}$ for each number field L. We scrutinize the behavior of $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/E_\infty}$ as an Iwasawa module for a fixed $\mathbb{Z}_p$-extension E_∞/E of a number field E and deduce an exact control theorem. A key ingredient of the proof is the isomorphism between the Pontryagin dual of the Selmer group and the module of Kähler differentials of the universal nearly ordinary deformation ring of φ. When $G = \mathrm{GL}(2)$, φ is a modular Galois representation and the base field E is totally real, from a recent result of Fujiwara identifying the deformation ring with an appropriate p-adic Hecke algebra, we conclude some fine results on the structure of the Selmer groups, including torsion-property and an exact limit formula at $s = 0$ of the characteristic power series, after removing the trivial zero.

TABLE OF CONTENTS

* Research of the author is partially supported by the NSF grant: DMS-9701017.

Received May 14, 1998

1. Introduction

Let $\mathcal{O}$ be a discrete valuation ring (with maximal ideal $\mathfrak{m} = \mathfrak{m}_{\mathcal{O}}$) finite flat over $\mathbb{Z}_p$ for a prime p. Let $G \subset \mathrm{GL}(n)$ be either $\mathrm{GL}(n)_{/\mathcal{O}}$ or a split similitude group defined over $\mathcal{O}$ by a symmetric or symplectic form. Let $\mathbb{J}$ be a local complete noetherian integral domain over $\mathcal{O}$ sharing the same residue field $\mathbb{F}$ with $\mathcal{O}$. Starting from a continuous Galois representation φ: $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to G(\mathbb{J})$ for a number field E, we let the Galois group act on the Lie algebra $\mathfrak{s}_{/\mathbb{J}}$ of the derived group S of G via adjoint action, getting the adjoint representation $\mathrm{Ad}_S(\varphi)$: $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to$

$\mathrm{GL}(\mathfrak{s})$. Suppose that φ factors through $\mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E)$ for the maximal extension $F^{(p,\infty)}/F$ unramified outside p and ∞, where F/E is a finite Galois extension F/E with $p \nmid [F:E]$. Further suppose that φ is nearly ordinary at p in the following sense: φ restricted to a decomposition group $D_\mathfrak{p}$ at a prime $\mathfrak{p}|p$ has values in a (proper) split parabolic subgroup $P_\mathfrak{p}$ of G. Thus φ restricted to $D_\mathfrak{p}$ leaves stable a flag $\mathcal{F} : 0 = V_0(\varphi) \subset V_1(\varphi) \subset \ldots \subset V_{m_\mathfrak{p}}(\varphi) = V(\varphi)$ of the space $V(\varphi)$ of φ. We write $\delta_{j,\mathfrak{p}}$ for the representation of $D_\mathfrak{p}$ on $V_j(\varphi)/V_{j-1}(\varphi)$. The Lie algebra $\mathfrak{n}_\mathfrak{p} \subset \mathfrak{s}$ of the unipotent radical of $P_\mathfrak{p} \cap S$ is stable under $D_\mathfrak{p}$. Now we can define, following Greenberg, the adjoint Selmer group $\mathrm{Sel}(\mathrm{Ad}_S(\varphi) \otimes \psi)_{/L}$ for any Artin representation $\psi\colon \Delta = \mathrm{Gal}(F/E) \to \mathrm{GL}(V) \cong \mathrm{GL}_m(\mathcal{O})$ and a subfield $L \subset F^{(p,\infty)}$ by

$$\begin{aligned}\mathrm{Sel}(\mathrm{Ad}_S(\varphi) \otimes \psi)_{/L} = \mathrm{Ker}(&H^1(\mathfrak{H}_L, (\mathfrak{s} \otimes_\mathcal{O} V)^*) \\ &\to \prod_{\mathfrak{p}|p} H^1(I_\mathfrak{p}, (\mathfrak{s} \otimes_\mathcal{O} V)^*/(\mathfrak{n}_\mathfrak{p} \otimes_\mathcal{O} V)^*)),\end{aligned}$$

where $\mathfrak{H}_L = \mathrm{Gal}(F^{(p,\infty)}/L)$, $I_\mathfrak{p}$ is the inertia subgroup at $\mathfrak{p}|p$ in $\mathfrak{H}_L$ and M^* for a $\mathbb{J}$-module M is given by $M \otimes_\mathbb{J} \mathbb{J}^*$ for the Pontryagin dual $\mathbb{J}^*$ of $\mathbb{J}$. Even though we have restricted ourselves to either $\mathrm{GL}(n)$ or similitude groups, our result actually covers any central extension $\pi : \tilde{G} \twoheadrightarrow G$, because we have $\mathrm{Ad}_S(\varphi) = \mathrm{Ad}_{\tilde{S}}(\tilde{\varphi})$ for any Galois representation $\tilde{\varphi}\colon \mathfrak{G} \to \tilde{G}(\mathbb{J})$ with $\varphi = \pi \circ \tilde{\varphi}$. In particular, our result is valid for metaplectic covers of GSp and spinor groups GSpin.

In this paper, we study the Iwasawa theory of $\mathrm{Sel}(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}$ for a fully p-ramified $\mathbb{Z}_p$-extension E_∞/E. We view the Pontryagin dual $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}$ of $\mathrm{Sel}(\mathrm{Ad}_S(\varphi) \otimes \psi)_{/E_\infty}$ as a module over the Iwasawa algebra $\mathbb{J}[[\Gamma]]$ for $\Gamma = \mathrm{Gal}(E_\infty/E) = \langle\gamma\rangle$ and study its module structure over $\mathbb{J}[[\Gamma]] = \mathbb{J}[[T]]$ $(T = \gamma - 1)$.

Since the group G is split reductive over $\mathcal{O}$, the conjugacy class of the split parabolic subgroup $P_\mathfrak{p}$ is represented by a unique **standard** parabolic subgroup $P^\circ_\mathfrak{p}$ containing a fixed (split) Borel subgroup B of G. The Borel subgroup B is determined by a unique (maximal) flag $\mathcal{F}$ of subspaces of $V(\varphi)$. Any standard parabolic subgroup is a stabilizer of a unique flag whose subdivision gives rise to $\mathcal{F}$. Therefore, if $P_\mathfrak{p} = g_\mathfrak{p} P^\circ_\mathfrak{p} g_\mathfrak{p}^{-1}$, then the flag $g_\mathfrak{p}\mathcal{F}$ is a subdivision of the flag $V_0(\varphi) \subset V_1(\varphi) \subset \ldots \subset V_{m_\mathfrak{p}}(\varphi)$. The Selmer group defined above depends on the choice of $P_\mathfrak{p}$ (and hence on the choice of the flag). For a given φ, there could be several choices of $P_\mathfrak{p}$. The choice of minimal $P_\mathfrak{p}$ would be a canonical one, although we do not care much in this paper which choice we make (except in our conjectures for which we make the hypothesis on φ that $P_\mathfrak{p}$ is a Borel subgroup). We may regard $\bigoplus_j \delta_{j,\mathfrak{p}}$ as a representation of $D_\mathfrak{p}$ into $M_\mathfrak{p}(\mathcal{O})$ for the Levi-quotient $M_\mathfrak{p} = P_\mathfrak{p}/N_\mathfrak{p}$, where $N_\mathfrak{p}$ is the unipotent radical of $P_\mathfrak{p}$.

In the above definition of the Selmer group, we have assumed that the ramification in φ outside p is rather limited to a finite extension F/E with $p \nmid [F:E]$, basically covering potentially unramified (outside p) cases. Thus we do not touch in this paper the case where some primes $\mathfrak{q} \nmid p$ semi-stably ramify in the representation φ. When G is bigger than GL(2), the primes $\mathfrak{q}$ ramify semi-stably in many different ways. Different types of ramification might require separate treatment. Since this paper inaugurates a general treatment of adjoint Selmer groups, we decided, at least for this paper, to avoid further complications coming from allowing infinite ramification outside p. We hope to come back to more general cases of ramification, if one can find anything new and interesting in more general cases.

If φ is the Galois representation of a pure motive M defined over E, we can define a motive $\mathrm{Ad}_S(M)$ to which $\mathrm{Ad}_S(\varphi)$ is associated. Its L-function is expected to satisfy the functional equation of the form $s \leftrightarrow 1-s$. If $\mathrm{Ad}(M)\otimes\psi$ is p-ordinary and is critical at $s=0$ and 1, then F is totally real and $P_\mathfrak{p}$ is a Borel subgroup. Suppose that M is regular (that is, all Hodge components of M have dimension ≤ 1). There is a good reason to believe that $L(1, \mathrm{Ad}_S(M)\otimes\psi) \neq 0$, which is true if φ is associated to a cuspidal automorphic representation of $\mathrm{GL}(n)_{/F}$. Thus if (i) $\mathrm{Ad}(M)\otimes\psi$ is critical, (ii) $P_\mathfrak{p}$ is a Borel subgroup and (iii) M is regular, the Selmer group $\mathrm{Sel}(\mathrm{Ad}(M)\otimes\psi)_{/E}$ should be finite, and we expect that the Pontryagin dual $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}$ is a torsion $\mathcal{O}[[T]]$-module of finite type, because its specialization at an arithmetic point tends to give $\mathrm{Sel}^*(\mathrm{Ad}(M)\otimes\psi)$ up to $\mathbb{J}$-torsion error (see (1) and (2) below). Even if F does have some complex places, we believe (the Pontryagin dual of) the Selmer group to be torsion as long as $P_\mathfrak{p}$ is a Borel subgroup and M is regular (see [H99a] Section 5). More generally, if $\mathrm{Ad}_S(\varphi)$ is arithmetic (that is, $\mathrm{Spec}(\mathbb{J})$ has densely populated points P such that $\mathrm{Ad}(\varphi) \bmod P$ is associated to a critical pure motive) and $P_\mathfrak{p}$ is a Borel subgroup (for all $\mathfrak{p}|p$), we expect that $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}$ is a torsion $\mathbb{J}[[T]]$-module of finite type.

Heuristically, if there exists a p-adic L-function $L_p(s, \mathrm{Ad}_S(M)\otimes\psi)$, the order of zero at $s=0$ should be equal to the number of linear Euler p-factors vanishing at $s=0$, because we expect to have $L(1, \mathrm{Ad}(M)\otimes\psi) \neq 0$ as described above. The number of such Euler factors can be computed as follows: Let $S_\mathfrak{p}$ be the set of primes of F over $\mathfrak{p}|p$ in E, on which Δ acts by conjugation. We consider the formal Δ-module $\mathbb{F}[S_\mathfrak{p}]$ generated by elements of $S_\mathfrak{p}$ over $\mathbb{F}$. Let $e_\mathfrak{p}(\psi)$ be the multiplicity of $\overline{\psi} = \psi \bmod \mathfrak{m}$ in $\mathbb{F}[S_\mathfrak{p}]$. Assuming that M is crystalline at $\mathfrak{p}$ and counting the multiplicity of 1 in the eigenvalues of the crystalline Frobenius, we conclude that the number of such linear Euler $\mathfrak{p}$-factors of $L(s, \mathrm{Ad}(M)\otimes\psi)$ should be $e_{\mathfrak{p}}(\psi)r_{\mathfrak{p}}$, where $r_{\mathfrak{p}}$ is the split rank of the center of the Levi-subgroup of $P_{\mathfrak{p}} \cap S$. Thus the number of such linear Euler p-factors should be $e = \sum_{\mathfrak{p}|p} e_{\mathfrak{p}}(\psi)r_{\mathfrak{p}}$ (see 4.3).

Under not so restrictive conditions we will describe later in this section, we shall prove the following assertions (see Section 4, in particular Theorem 4.4):

(1) We have, if $e = 0$,

$$\frac{\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}}{T\,\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}} \cong \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E};$$

(2) In general, we have the following exact sequence:

$$\mathbb{J}^e \xrightarrow{\iota_I} \frac{\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}}{T\,\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}} \to \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E} \to 0.$$

We conjecture that ι_I is injective (Conjecture 4.2) and that $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/E_\infty}$ is pseudo-isomorphic to a product of $\mathbb{J}^e$ and a torsion $\mathbb{J}[[T]]$-module M without trivial zero at $T = 0$, as long as $P_{\mathfrak{p}}$ is a Borel subgroup and $\mathrm{Ad}_S(\varphi)$ is arithmetic ([H97a]). This arithmeticity essentially forces φ to have values in $\mathrm{GSp}(n)$ or $\mathrm{GO}(n)$ if $n > 2$ (see Example 2.7 and Example 2.8).

A sufficient condition to have control with finite error, similar to (1), has been given by Ochiai for general crystalline Galois representations without trivial zero (that is, the crystalline Frobenius does not have the eigenvalue 1; see [O]) by a different method. For example, one starts with a 2-dimensional representation $\varphi\colon \mathfrak{G} \to \mathrm{GL}_2(\mathcal{O})$ and makes a symmetric power $\mathrm{Sym}^k\varphi\colon \mathfrak{G} \to \mathrm{GL}_{k+1}(\mathcal{O})$. Ochiai's result gives a control theorem (with finite bounded error) for $\mathrm{Sel}(\det(\varphi)^j\,\mathrm{Sym}^k(\varphi))$ for odd k, when φ is associated to a critical 2-dimensional pure motive ordinary at p. On the other hand, since $\mathrm{Ad}_{\mathrm{SL}(k+1)}(\mathrm{Sym}^k(\varphi)) \cong \bigoplus_{j=1}^k \varphi_j$ for $\varphi_j = \det(\varphi)^{-j}\,\mathrm{Sym}^{2j}(\varphi)$, our result in essence takes care of symmetric even powers of φ (see Examples 4.1 and 6.2). This work of Ochiai also deals with the Selmer groups of Bloch–Kato for critical crystalline motives M satisfying the Panchishkin condition.

Suppose $G = \mathrm{GL}(2)$, F is totally real and $\mathrm{Spec}(\mathbb{J})$ is an irreducible closed subscheme of $\mathrm{Spec}(h^{n.ord})$ for the universal nearly ordinary Hecke algebra $h^{n.ord}$ for $\mathrm{GL}(2)_{/F}$. Then, under suitable assumptions, the Hecke algebra represents the (nearly ordinary) deformation functor deforming the representation $\overline{\rho}_F = \varphi \bmod \mathfrak{m}$ of $\mathfrak{H}_F$. This follows from a recent work of Fujiwara generalizing an earlier work of Wiles–Taylor and Diamond for $F = \mathbb{Q}$ (see Section 5). From this, we can conclude (Theorems 6.1 and 6.3), under suitable assumptions, that

(I) $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi) \otimes \psi)_{/E}$ is a torsion $\mathbb{J}$-module of finite type and has no pseudo-null $\mathbb{J}$-module non-null;

(II) The map ι_I is injective;

(III) $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi) \otimes \psi)_{/E_\infty}$ is a torsion $\mathbb{J}[[T]]$-module of finite type;

(IV) If $e = 0$ and $\mathbb{J}$ is a regular local ring, then $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi) \otimes \psi)_{/E_\infty}$ has no pseudo-null $\mathbb{J}[[T]]$-module non-null.

Let O be the integer ring of F, and we put $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then we write $I_0 \subset O_p^\times$ for the subgroup made of universal norms from F_∞/F ($F_\infty = E_\infty F$). Then $O_p^\times/I_0 \cong \Gamma^{S_F}$ for the set S_F of all primes of F over p. We number elements in S_F as $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ for $s = |S_F|$. We write $\mathrm{Cl}(p^\infty)$ for the Galois group of maximal abelian extension of F unramified outside p and ∞. Then $h^{n.ord}$ is an algebra over the Iwasawa algebra $\mathcal{O}[[\mathbf{G}]]$ for $\mathbf{G} = \mathrm{Cl}(p^\infty) \times O_p^\times$. Therefore, if we write $\mathcal{O}[[\mathbf{G}]] = \mathcal{O}[[\mathrm{Cl}(p^\infty) \times I_0]][[T_1, \ldots, T_s]]$ for parameters T_j with $s = |S_F|$, we can think of the jacobian determinant

$$\mathbf{J} = \det\left(\frac{\partial \mathbb{T}(\mathfrak{p}_i)}{\partial T_j}\right)$$

in $h^{n.ord}$ for Hecke operators $\mathbb{T}(\mathfrak{p}_i)$. If $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(h^{n.ord})$, the existence of the conjectural pseudo-isomorphism of the Selmer group into $\mathbb{J}^e \times M$ as above is equivalent to the non-vanishing of the image J of $\mathbf{J}$ in $\mathbb{J}$. If $\mathbb{J}$ is a regular local ring, assuming that $J \neq 0$, we shall show that

(V) $$\Psi(T) = T^e \Phi(T) \quad \text{with } \Phi(0) = J\eta \in \mathbb{J} \text{ up to units}$$

for the characteristic element $\eta \in \mathbb{J}$ of $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi) \otimes \psi)_{/E}$ and for the characteristic power series $\Psi(T) \in \mathbb{J}[[T]]$ of $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi) \otimes \psi)_{/E_\infty}$ (Theorem 6.3). We also prove the non-vanishing: $J \neq 0$ for the cyclotomic $\mathbb{Z}_p$-extension F_∞/F if $\mathrm{Spec}(\mathbb{J})$ is sufficiently large and contains a point P whose Galois representation $\varphi \bmod P$ is of multiplicative type at all but one p-adic place (Proposition 7.1).

To prove the above assertions, we have exploited an idea of Mazur, that is, the identification of the Selmer group with the module of Kähler differentials of the global universal deformation space of $\overline{\rho}_L = \varphi|_{\mathfrak{H}_L} \bmod \mathfrak{m}_{\mathbb{J}}$ over the local one (Theorem 2.3), where $\mathfrak{m}_{\mathbb{J}}$ is the maximal ideal of $\mathbb{J}$. Here we call a representation $\rho\colon \mathfrak{H}_L \to G(A)$ for an Artinian local $\mathcal{O}$-algebra A with residue field $\mathbb{F}$ a deformation of $\overline{\rho}_L$ if $\rho \equiv \overline{\rho}_L \bmod \mathfrak{m}_A$ for the maximal ideal $\mathfrak{m}_A$ of A. Thus we need to assume the representability of the deformation functors, which follows from the following conditions:

(Z_F) For any deformation $\rho\colon \mathfrak{H}_F \to G(A) \subset \mathrm{GL}_n(A)$ of $\overline{\rho}_F$, if $x\rho = \rho x$ for $x \in \mathrm{GL}_n(A)$, then x is scalar;

($Z_{p,F}$) For any deformation $\delta: I_{\mathfrak{p}} \to \mathrm{GL}_m(A)$ (for all $\mathfrak{p}|p$) of $\overline{\delta}_{j,\mathfrak{p}} = \delta_{j,\mathfrak{p}} \bmod \mathfrak{m}_{\mathbb{J}}$, if $x\delta = \delta x$ for $x \in \mathrm{GL}_m(A)$, x is scalar, where $I_{\mathfrak{p}}$ is the inertia subgroup of $\mathfrak{H}_F$;

(Reg_F) $H^0(D_{\mathfrak{p}}, \mathfrak{s}/(\mathfrak{s} \cap \mathcal{P}_{\mathfrak{p}})) = 0$ for the Lie-algebra $\mathcal{P}_{\mathfrak{p}}$ of $P_{\mathfrak{p}}$ for all prime $\mathfrak{p}|p$.

The conditions (Z_F) and ($Z_{p,F}$) follows from the absolute irreducibility of the corresponding $\mathbb{F}$-residual representation, but there are many examples of residual representations satisfying these conditions which are not absolutely irreducible. Similarly, (Reg_F) follows from $\mathrm{Hom}_{D_{\mathfrak{p}}}(\overline{\delta}_{i,\mathfrak{p}}, \overline{\delta}_{j,\mathfrak{p}}) = 0$ for all $\mathfrak{p}$ and $i \neq j$. We need to assume one more technical assumption for the validity of the assertions (1) and (2):

$$\text{(EP)} \qquad p \nmid 2n[F:E] \prod_{\mathfrak{p}} \prod_{j=1}^{m_{\mathfrak{p}}} \dim \overline{\delta}_{j,\mathfrak{p}}.$$

At this moment, for the validity of Fujiwara's result for $\mathrm{GL}(2)_{/F}$ quoted above, we need to assume the following conditions:

(AI_k) $\overline{\rho}_k: \mathfrak{H}_k \to \mathrm{GL}_2(\mathbb{F})$ is absolutely irreducible for $k = F(\sqrt{(-1)^{(p-1)/2}p})$;

(LD) F is linearly disjoint from $\mathbb{Q}(\mu_p)$ over $\mathbb{Q}$, and $O_p^{\times}$ is p-torsion-free;

(NR) $F_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified if $\overline{\rho}$ is flat at $\mathfrak{p}$.

Thus, for the assertions stated above concerning $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(h^{n.ord})$, we need to assume ($\mathrm{AI}_k$), (LD) and (NR) (or simply the universality of $h^{n.ord}$: (univ) in 5.2). The condition (AI_k) follows from the absolute irreducibility of $\mathrm{Ad}_{\mathrm{SL}(2)}(\overline{\rho}_F)$. Since $S \subset G$ is simple, $\mathrm{Ad}_S(\overline{\rho}_F)$ is absolutely irreducible for a representation $\overline{\rho}: \mathfrak{H}_F \to G(\mathbb{F})$ with sufficiently large image.

Since Fujiwara's formulation of the method of Taylor–Wiles is formal, we expect its generalizations to more general groups G in near future (some cases have already been dealt with by Harris–Taylor [HaT]). Once we have the identification of the (nearly ordinary) Hecke algebra of the Langlands dual G^L of G with an appropriate global p-adic deformation ring of $\overline{\rho}_F$, we should be able to get statements similar to I–V (from (1)–(2)) for more general G rather than just $\mathrm{GL}(2)_{/F}$. This is the reason why we have treated general split classical groups G.

When $F = \mathbb{Q}$, the universal ordinary Hecke algebra h^{ord} is finite flat over $\mathcal{O}[[W]]$ for the weight variable W, and $h^{n.ord} = h^{ord}[[T]]$. When $\mathbb{J}$ is an irreducible component of $\mathrm{Spec}(h^{ord})$, we constructed in [H90] a p-adic L-function $L_p(W,T) \in T\mathbb{J}[[T]]$ associated to $\mathrm{Ad}_{\mathrm{SL}(2)}(\varphi)$ and the $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$. Hence, we have a main conjecture asserting the identity: $L_p(W,T) = \Psi(T)$ for $\Psi(T)$ as above.

Greenberg and Tilouine have shown the congruence

$$\frac{L_p}{T}(0,0) \equiv J\eta \bmod P$$

if an arithmetic prime P of $\mathbb{J}$ is associated to an elliptic curve with multiplicative reduction at p. Under some (standard) conjectures on Siegel modular varieties with some addtional hypothses, Urban has shown the divisibility: $L_p|\Psi$; so, the main conjecture: $\Psi = L_p$ follows from the assertion (V) and the non-vanishing: $J \not\equiv 0 \bmod P$ ([BDGP]) in this (elliptic curve) case (see [HTU] and [U] for more details). After the solution of this two-variable main conjecture, the one-variable version for an elliptic Hecke eigen cusp form f automatically follows, because $s \mapsto L_p(w, \gamma^s - 1)$ for a suitable specialization $W \mapsto w \in \mathcal{O}$ gives the canonical (or **genuine** in the terminolgy of [H97b]) p-adic L-function $L_p(s, \mathrm{Ad}(f))$ of $\mathrm{Ad}(f)$. This canonical p-adic L-function differs by a constant η_f from the cyclotomic p-adic L-function for $\mathrm{Ad}(f)$ constructed by other authors, if f has non-trivial congruence modulo p with other elliptic cusp forms. Here η_f is the order of the congruence module of f. There is no doubt that the construction in [H90] generalizes to the Hilbert modular case.

In Section 2, we describe various Selmer groups and study relations among them. In Section 3, we study base-change of the deformation rings, and in Section 4, we prove the assertions (1) and (2). In Section 5, we recall basic facts from the theory of Hecke algebras for GL(2) and deduce the universality of the Hecke algebra from the result of Fujiwara [Fu]. The assertions I–V will then be proven in Section 6. In Section 7, we prove the non-vanishing: $J \neq 0$ in almost multiplicative reduction case. At the end, we shall give corrections to the result in [H96a], although we do not use in this paper the assertion mis-stated there.

This paper supersedes my earlier preprints [H97a] and [H97b]. The principal idea of this paper is similar to the idea described in [H97a] for $F = \mathbb{Q}$ and $G = \mathrm{GL}(2)$ (which is not for publication, except for some parts reproduced in [H99b] Chapter 5 and is slightly different from the method of [H96a]). The idea has been fully developed in this paper to include classical split groups G. After having written [H97b], I realized that the argument works well for general classical groups G and symmertic even powers φ_j, and this paper is the outcome of the endeavor.

LIST OF SYMBOLS.

Section 1

$\mathcal{O}$, $\mathbb{F}$: the beginning of Section 1; 2.3
$\Psi(T)$: the assertion (V)

Section 2

$F^{(p,\infty)}/F$: 2.1
$\mathrm{Sel}_{\mathcal{S}}(\varphi)_{/L}$: 2.1, (Sel)
$\mathcal{S} = \{V(\delta_{\mathfrak{p}}^{-})\}$: Selmer datum, 2.1
S_E: 2.1, above ($\mathrm{fil}_{\mathfrak{p}}$)
$\delta_{j,\varphi,\mathfrak{p}}$: 2.1, ($\mathrm{fil}_{\mathfrak{p}}$)
$P_{\mathfrak{p}}$: 2.1, ($\mathrm{fil}_{\mathfrak{p}}$); 2.3, after (Sm)
$\delta_{\mathfrak{p}}^{\pm}$: 2.1, (Sel)
$\mathrm{Ad}(\varphi)$: 2.3
$V_{\mathfrak{p}}^{\pm}(\mathrm{Ad}(\varphi))$, $V_{\mathfrak{p}}^{\pm}(\mathrm{Ad}_{\mathcal{S}}(\varphi))$: 2.3, (+), (−), (±)
$\mathrm{Sel}(\mathrm{Ad}(\varphi))$, $\mathrm{Sel}_{-}(\mathrm{Ad}(\varphi))$: 2.3
$\mathrm{Ad}(M)$: 2.3, 2.7
$\mathrm{Ad}_{\mathcal{S}}(\varphi)$: 2.3, (2.1)
$\phi = (\phi_{\det}, \phi_{\nu})$: 2.3, after (2.1)
(AI_L), (Reg_L), (RG_L), (Z_L), ($\mathrm{Z}_{p,L}$): 2.4
type $\mathcal{F}$: 2.4 above ($\mathrm{fil}_{\mathfrak{p}}$)
$\Phi_{G,L}^{\phi}$: 2.4, below ($\mathrm{fil}_{\mathfrak{p}}$)
$(R_{G,F}^{\phi}, \varrho_{G,F}^{\phi})$: 2.4, below ($\mathrm{fil}_{\mathfrak{p}}$)
$m(\psi)$: 2.6, above (TW)
$X[\psi]$: 2.6, above (TW)
$V_{\mathfrak{p}}^{-}(\varphi_E \otimes \psi)$: 2.6, (2.15)
$\mathrm{Sel}(\varphi_E \otimes \psi)_{/L}$: 2.6, (2.18)
$\mathrm{Sel}(\mathrm{Ind}_F^E \mathrm{Ad}(\varphi_F))_{/L}$: 2.6, (2.19)

Section 3

$\Phi_{G,L}^{?}$, $(R_{G,L}^{?}, \varrho_{G,L}^{?})$: 3.1
$\alpha^{?} = \alpha_{L/L'}^{?}$: 3.2

Section 4

(TR): Section 4 above 4.1
$R_{j,\bar{\delta}_{i,\mathfrak{p}}}^{H}$, R_j^H: 4.1, 4.2
$m_{\mathfrak{p}}$: 4.2

Section 5

$U_0(N)$, $U_{11}(N)$, $U_1(N)$: 5.1, (5.1)
$L(n,v;A)$, $L(n,v,\chi;A)$: 5.1, (5.4)
$S(U;A)$: 5.1, (5.6)
$H_{cusp}^{d}(X(U),L)$: 5.1, (5.7)
$\mathbf{G}$: 5.1, (5.8)
$U_{\alpha}^{?}$, U_{α}^{W}: 5.1, (5.10), (5.11)
$\mathbb{G}_p^{?}$, $\mathbf{G}_p^{?}$: 5.1, below (κ)
$h_{\kappa}^{?}(\mathcal{O})$: 5.1, above ($\kappa$)
$\kappa = (n, v, \varepsilon, \xi)$: 5.1, below (Hecke)
$M^{?}$: 5.1, (ff)
local ring $h^{?}$: 5.1, above (rep)
κ_0: at the beginning of 5.2
h^{κ}: at the beginning of 5.2
I_F: 5.2, above (NR_p)
(cpi), (univ): 5.2, below Theorem 5.1
ϕ_{κ}, Φ_{κ}: 5.2, Remark 5.1

Section 6

D_L, I_L, D_j, I_j: 6.2
$t_{\mathfrak{p}}$, ϖ_j: 6.2, around Conjecture 6.2
J: 6.2, Conjecture 6.2
η, $\Phi(T)$: 6.2, Theorem 6.3

2. Various adjoint Selmer groups

2.1. DEFINITION. We begin with the definition by Greenberg of the Selmer groups we like to study. Let F/E be a Galois extension of number field inside a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. We fix a prime $p \nmid [F:E]$. We write $F^{(p,\infty)}/F$ for the maximal extension of F unramified outside $\{p,\infty\}$. Let $\mathbb{J}$ be a p-adic pro-artinian local $\mathcal{O}$-algebra with finite residue field $\mathbb{F}$, and we take a Galois representation φ: $\mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E) \to \mathrm{GL}_n(\mathbb{J})$. We write $V = V(\varphi)$ for the representation space of φ, which is a $\mathbb{J}$-free module of rank n. Let S_E be the set of prime factors of p in E. For each $\mathfrak{p} \in S_E$, we fix a decomposition subgroup $D_\mathfrak{p}$ in $\mathfrak{G}$. We assume that we are given a $D_\mathfrak{p}$-stable filtration:

$$(\mathrm{fil}_\mathfrak{p}) \qquad 0 = V(\varphi)_{0,\mathfrak{p}} \subset V(\varphi)_{1,\mathfrak{p}} \subset V(\varphi)_{2,\mathfrak{p}} \subset \cdots \subset V(\varphi)_{m_\mathfrak{p},\mathfrak{p}} = V(\varphi),$$

where we assume that $V(\varphi)/V(\varphi)_{j,\mathfrak{p}}$ for all $j = 1, \ldots, m_\mathfrak{p} - 1$ are all $\mathbb{J}$-free. The stabilizer of this filtration gives rise to a conjugacy class of a parabolic subgroup $P_\mathfrak{p}$ of $\mathrm{GL}(n)$. We call the representation φ nearly ordinary of type $\mathcal{F} = \{P_\mathfrak{p}\}_\mathfrak{p}$ and call $\mathcal{F}$ the nearly ordinarity datum for φ. When all $P_\mathfrak{p}$ are Borel subgroups, we call φ (nearly ordinary) of Borel type. We write $\delta_{j,\varphi,\mathfrak{p}}$ for the representation of $D_\mathfrak{p}$ on $V(\varphi)_{j,\mathfrak{p}}/V(\varphi)_{j-1,\mathfrak{p}}$. If φ is of Borel type, $\delta_{j,\varphi,\mathfrak{p}}$ is a character.

We fix one step $0 \le j \le m$. We write $\delta_\mathfrak{p}^-$ for the representation of $D_\mathfrak{p}$ on $V(\varphi)_{j,\mathfrak{p}} = V(\delta_\mathfrak{p}^-)$. We call such a datum a **local Selmer datum** $\mathcal{S} = \{V(\delta_{\varphi,\mathfrak{p}}^-)\}_{\mathfrak{p}\in S}$. Then we consider the Pontryagin dual $\mathbb{J}$-module

$$\mathbb{J}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{J}, \mathbb{Q}_p/\mathbb{Z}_p).$$

For each $\mathbb{J}$-module X, we define $X^* = X \otimes_\mathbb{J} \mathbb{J}^*$ and let $\mathfrak{G}$ or its subgroup act on X^* through the left factor if X has an action of the subgroup. We consider the Galois cohomology group $H^1(\mathfrak{G}, V(\varphi)^*)$, which is a discrete $\mathbb{J}$-torsion module. Then we define, writing $\mathfrak{H}_L = \mathrm{Gal}(F^{(p,\infty)}/L)$ for each intermediate extension $F^{(p,\infty)}/L/E$, the **Selmer group** with respect to $\mathcal{S}$ by

$$(\mathrm{Sel}) \qquad \mathrm{Sel}_\mathcal{S}(\varphi)_{/L} = \mathrm{Ker}(H^1(\mathfrak{H}_L, V(\varphi)^*) \to \prod_{\mathfrak{p}\in S_L} H^1(I_\mathfrak{p}, V(\delta_\mathfrak{p}^+)^*)),$$

where $V(\delta_\mathfrak{p}^+) = V(\varphi_\mathfrak{p})/V(\delta_{\varphi,\mathfrak{p}}^-)$, $I_\mathfrak{p}$ is the inertia subgroup of the decomposition group $D_\mathfrak{p} \subset \mathfrak{H}_L$ and the map is the restriction composed with the projection: $V(\varphi)^* \to V(\delta_{\varphi,\mathfrak{p}}^+)^*$. We can define the **strict** Selmer group $\mathrm{Sel}_{\mathcal{S},st}(\varphi)_{/L}$ replacing $I_\mathfrak{p}$ in the above definition by $D_\mathfrak{p}$.

Example 2.1: Let $F/\mathbb{Q}$ be a quadratic extension (so $E = \mathbb{Q}$) and $\chi : \mathrm{Gal}(F/\mathbb{Q}) \cong \{\pm 1\}$ be the unique non-trivial character. We may regard χ as having values in $\mathbb{Z}_p^\times$ for $p > 2$. We take $V(\delta_p^+)$ to be the full space $V(\chi)$. Then $\mathrm{Sel}(\chi)_{/\mathbb{Q}}$ is isomorphic to the p-part $\mathrm{Cl}_{F,p}$ of the strict class group of F. This can be proven as follows: By Inflation-restriction sequence, we have

$$H^1(\mathfrak{G}, V(\chi)^*) \cong H^1(\mathfrak{H}, V(\chi)^*)[\chi] \cong \mathrm{Hom}_{\mathfrak{G}}(\mathfrak{H}, V(\chi)^*),$$

where "$[\chi]$" indicates the χ-eigenspace. Then the above isomorphism induces

$$\mathrm{Sel}(\chi) \overset{\alpha}{\cong} \mathrm{Hom}(\mathrm{Cl}_{F,p}, V(\chi)^*) \cong \mathrm{Cl}_{F,p}.$$

The isomorphism α follows from class field theory, and the last isomorphism holds because $\mathrm{Cl}_{F,p}[\mathrm{id}]$ is trivial (that is, $\mathbb{Z}$ is a PID!). More generally, taking a finite cyclic extension $F/\mathbb{Q}$ of degree n with $p \nmid n$ and let χ: $\mathfrak{G} \cong \mu_n \subset \mathcal{O}^\times$ be a character such that $F = L^{\mathrm{Ker}(\chi)}$ for $L = F^{(p,\infty)}$. Then for the choice: $V(\delta_p^+) = V(\chi)$, the Selmer group $\mathrm{Sel}(\chi)_{/\mathbb{Q}}$ is isomorphic to the χ-eigenspace of $\mathrm{Cl}_{F,p} \otimes_{\mathbb{Z}_p} \mathcal{O}$.

Example 2.2: Let $E = \mathbb{Q}$. Let $\mathcal{E}_{/\mathbb{Q}}$ be an elliptic curve with ordinary good reduction at $p > 2$. We suppose that $\mathcal{E}$ acquires everywhere good reduction for a finite extension F with $p \nmid [F : \mathbb{Q}]$. The Tate module $T_p(\mathcal{E})$ has a natural filtration:

$$0 \to T_p(\mathcal{E}^c) \to T_p(\mathcal{E}) \to T_p(\mathcal{E}^{et}) \to 0.$$

Here $\mathcal{E}^c$ (resp. $\mathcal{E}^{et}$) is the connected component (resp. the maximal étale quotient) of the p-divisible group of $\mathcal{E}$. We take $V(\varphi)$ to be $T_p(\mathcal{E})$ and $V(\delta_p^+)$ to be $T_p(\mathcal{E}^{et})$. We write $\mathrm{Sel}(\mathcal{E})_{/L}$ for $\mathrm{Sel}(\varphi)_{/L}$ for this φ. Then by Kummer theory for $\mathcal{E}$, we have an exact sequence:

$$0 \to \mathcal{E}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}(\mathcal{E})_{/\mathbb{Q}} \to \text{Ш}(\mathcal{E})_{/\mathbb{Q}} \to 0.$$

Here $\text{Ш}(\mathcal{E})_{/\mathbb{Q}}$ is the p-primary part of the Tate–Shafarevich group for $\mathcal{E}$ over $\mathbb{Q}$. We refer details for this type of Selmer groups to Greenberg's exposition [Gr1] Section 2.

Example 2.3: Let $f \in S_k(\Gamma_0(C), \chi)$ be a Hecke eigenform for a "Neben" Dirichlet character χ of conductor C. Write $f|T(n) = \lambda(T(n))f$ for a system of Hecke eigenvalues $\lambda(T(n))$, and let $\mathbb{Q}(\lambda)$ be the number field generated by $\lambda(T(n))$ for all n. Take a prime ideal $\mathfrak{p}|p$ of $\mathbb{Q}(\lambda)$, and let $\mathcal{O}$ be the $\mathfrak{p}$-adic integer ring. Then we have a Galois representation φ: $\mathfrak{G} \to \mathrm{GL}_2(\mathcal{O})$ associated to λ characterized by the fact: $\mathrm{Tr}(\varphi(\mathrm{Frob}_\ell)) = \lambda(T(\ell))$ for all primes $\ell \nmid Cp$. We further suppose that $\lambda(T(p)) \in \mathcal{O}^\times$. Then $V(\varphi)$ has a natural filtration:

$$0 \to V(\delta_p^-) \to V(\varphi) \to V(\delta_p^+) \to 0$$

stable under D_p (cf. [MW] and [Ti]). Here $\mathrm{rank}_{\mathcal{O}} V(\delta_p^\pm) = 1$. We suppose that the order of χ is prime to $p > 2$. Then we take $E = \mathbb{Q}$ and F to be the cyclic extension of $\mathbb{Q}$ such that $\widehat{\chi} : \mathrm{Gal}(F/\mathbb{Q}) \cong \mathrm{Im}(\chi)$ for the Galois character $\widehat{\chi}$ associated to χ by class field theory. We can think of $\mathrm{Sel}(\varphi)_{/\mathbb{Q}}$ as in the previous example, but here we look into $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$. We let $\mathfrak{G}$ act on $M_2(\mathcal{O}) = \mathrm{End}_{\mathcal{O}}(V(\varphi))$ by conjugation. Then the subspace $V(\mathrm{Ad}(\varphi)) \subset M_2(\mathcal{O})$ made of trace zero matrices is stable under the action. In this way, we get a three dimentional representation $\mathrm{Ad}(\varphi)$. The one dimensional subspace

$$W = \left\{\phi \in \mathrm{End}_{\mathcal{O}}(V(\varphi)) \middle| \ \mathrm{Tr}(\phi) = \phi(V(\delta_p^-)) = 0\right\} \cong \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\} \subset V(\mathrm{Ad}(\varphi))$$

is then stable under D_p. We define the Selmer group $\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ taking $V(\delta^+_{\mathrm{Ad}(\varphi),p})$ to be $V(\mathrm{Ad}(\varphi))/W$. This Selmer group has been studied in depth by Wiles in [W], and Wiles' work combined with an earlier result of mine yields:

$$|\,\mathrm{Sel}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}| = \left|\frac{\Gamma(1,\mathrm{Ad}(\lambda))L(1,\mathrm{Ad}(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}\right|_p^{-1}.$$

Here $A = \mathbb{Q}(\lambda) \cap \mathcal{O}$ is the discrete valuation ring of $\mathbb{Q}(\lambda)$ (induced by $\mathcal{O}$), the $\pm$-periods $\Omega(\pm,\lambda;A)$ are the normalized $\pm$-periods of f with respect to A, and $L(s,\mathrm{Ad}(\lambda)) = L(s,\mathrm{Ad}(\varphi))$ is the adjoint L-function of λ with Γ-factor $\Gamma(s,\mathrm{Ad}(\lambda))$. We refer to [H99b] Chapter V Section 3, [H96b] Section 2.9-10 and [DHI] the details of these periods and the L-function.

Example 2.4: We can think of a more general setting than the above examples. Suppose that $F = \mathbb{Q}$, $\mathbb{J} = \mathbb{Z}_p$ and that φ is the etale realization of a rank n pure motive $M_{/\mathbb{Q}}$ crystalline at p. We suppose that M is critical. Thus there is the middle term of the Hodge filtration $\mathcal{F}^-(M) \subset H_{DR}(M)$. We then assume that $V(\delta_p^-)$ is sent onto $\mathcal{F}^-(M)$ by the p-adic comparison isomorphism. If $L(0,M) \neq 0$, we expect that $|\,\mathrm{Sel}(\varphi)|$ is finite and is related to $|\frac{L(0,M)c_p^+(M)}{c_\infty^\pm(M)}|_p^{-1}$, where $c_?^+(M)$ is the period normalized with respect to $V(\delta_p^-)$ and $\mathcal{F}^-(M)$ (see [H96b] Chapter 3). Under the notation of Example 2.3, $V(\varphi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is the p-adic étale realization of a rank two motive M of Hodge weight $(k-1,0)$ and $(0,k-1)$. We can then split $M \otimes M^\vee = \mathrm{Ad}(M) \oplus \mathbf{1}$ for the rank 3 adjoint motive $\mathrm{Ad}(M)$, where $M^\vee$ is the dual of M. Then the subspace W in Example 2.3 corresponds to $\mathcal{F}^+(\mathrm{Ad}(M)) =$

$\mathcal{F}^-(\mathrm{Ad}(M)(1))$ under the p-adic comparison isomorphism (see Example 2.7 and [H96b] Section 3.2). Then $c_\infty^+(\mathrm{Ad}(M)(1)) = \Omega(+,\lambda;A)\Omega(-,\lambda;A)$ up to elements in $\mathbb{Q}(\lambda)^\times$ (and a power of $(2\pi i)$ to cancel the $(2\pi i)$ from the Γ-factor). The Ω-periods are normalized so that $c_p^+(\mathrm{Ad}(M)(1))$ is a p-adic unit.

2.2. CHARACTER TWISTS. Let E_∞/E be a $\mathbb{Z}_p$-extension with $\Gamma = \mathrm{Gal}(E_\infty/E) \cong \mathbb{Z}_p$. We have the tautological character $\kappa : \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]^\times$ for the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. Fix a generator $\gamma \in \Gamma$. Then we can identify $\mathbb{J}[[\Gamma]]$ with a power series ring $\mathbb{J}[[T]]$ so that $\kappa(\gamma) = 1+T$. We regard κ as a character of $\mathfrak{G}$ and then consider $\varphi \otimes \kappa$: $\mathrm{Gal}(F^{(p,\infty)}/E) \to \mathrm{GL}_n(\mathbb{J}[[\Gamma]])$. We want to relate $\mathrm{Sel}(\varphi\otimes\kappa)_{/E}$ with respect to $\{V(\delta_\mathfrak{p}^-\otimes\kappa)\}_\mathfrak{p}$ and $\mathrm{Sel}(\varphi)_{/E_\infty}$ with respect to $\{V(\delta_\mathfrak{p}^-)\}_\mathfrak{p}$. As proved in [H96a] Section 3.1, we have

PROPOSITION 2.1: *We have* $\mathrm{Sel}(\varphi\otimes\kappa)_{/E} \cong \mathrm{Sel}(\varphi)_{/E_\infty}$.

Thus we may consider $\mathbb{J}[[\Gamma]]$ as the coefficient ring of $\mathrm{Sel}(\varphi\otimes\kappa)_{/E}$, and Greenberg has conjectured that $\mathrm{Sel}^*(\varphi\otimes\kappa)_{/E}$ is a torsion module over its coefficient ring $\mathbb{J}[[\Gamma]]$ if the associated p-adic L-function does not vanish ([Gr] Conjecture 4.1).

Example 2.5: We keep the notation of Example 2.1. Thus χ is a character inducing $\mathrm{Gal}(F/\mathbb{Q}) \cong \mu_n \subset \mathcal{O}^\times$. Let $E_\infty = \mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension. We suppose that χ is an odd character. We write $\kappa : \Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \hookrightarrow \mathcal{O}[[\Gamma]]$ for the tautological character. We then regard $\chi\kappa$ as a character of $\mathfrak{G}$ having values in $\mathcal{O}[[\Gamma]]^\times$. Then taking $V(\delta_p^+)$ to be the full space $V(\chi\kappa)$, we have $\mathrm{Sel}(\chi\kappa)_{/\mathbb{Q}} \cong \mathrm{Sel}(\chi)_{/\mathbb{Q}_\infty}$. The Pontryagin dual $\mathrm{Sel}^*(\chi)_{/\mathbb{Q}_\infty}$ is the classical Iwasawa module studied by Iwasawa (see Introduction of [Gr1]).

Example 2.6: We keep the notation in Example 2.2. The study of the Selmer groups $\mathrm{Sel}(\mathcal{E})_{/\mathbb{Q}_\infty}$ (and $\mathrm{Sel}(\mathcal{E})_{/E_\infty}$) was initiated by Mazur (see [Gr1]). In particular, if $\mathcal{E}$ is modular (that is now known to be true for almost all rational elliptic curves by Wiles and others), Mazur constructed a p-adic L-function of $\mathcal{E}_{/\mathbb{Q}}$ (see, for example, [H93] Chapter 6), and he conjectured that the characteristic power series of $\mathrm{Sel}(\mathcal{E})_{/\mathbb{Q}_\infty}$ is given by the p-adic L-function ([Gr1] Conjecture 1.13).

2.3. ADJOINT GALOIS REPRESENTATION. We now let $\mathfrak{G}$ act on $M_n(\mathbb{J})$ by conjugation: $x \mapsto \varphi(\sigma)x\varphi(\sigma)^{-1}$. The trace zero subspace $\mathfrak{sl}$ is stable under this action. This new Galois module of dimension n^2-1 is called the adjoint representation of φ and written as $\mathrm{Ad}(\varphi)$. Thus

$$V = V(\mathrm{Ad}(\varphi)) = \Big\{T \in \mathrm{End}_\mathbb{J}(V(\varphi)) \Big| \ \mathrm{Tr}(T) = 0\Big\}.$$

This space has a three step filtration: $0 \subset V_{\mathfrak{p}}^{+} \subset V_{\mathfrak{p}}^{-} \subset V$ given by

$$(+) \quad V_{\mathfrak{p}}^{+}(\mathrm{Ad}(\varphi)) = \left\{ T \in V(\mathrm{Ad}(\varphi)) \middle| \, T(V(\varphi)_{j,\mathfrak{p}}) \subset V(\varphi)_{j-1,\mathfrak{p}} \text{ for all } j \right\},$$

$$(-) \quad V_{\mathfrak{p}}^{-}(\mathrm{Ad}(\varphi)) = \left\{ T \in V(\mathrm{Ad}(\varphi)) \middle| \, T(V(\varphi)_{j,\varphi,\mathfrak{p}}) \subset V(\varphi)_{j,\varphi,\mathfrak{p}} \text{ for all } j \right\}.$$

From this, we can think of four different Selmer groups for each $\mathfrak{p}$ depending on $\mathcal{S}$, which should correspond different p-adic L-functions. We simply write

$$\mathrm{Sel}_0(\mathrm{Ad}(\varphi)), \quad \mathrm{Sel}(\mathrm{Ad}(\varphi)), \quad \mathrm{Sel}_{-}(\mathrm{Ad}(\varphi)), \quad \text{or} \quad \mathrm{Sel}_{full}(\mathrm{Ad}(\varphi))$$

according as $\mathcal{S} = 0$, $V_{\mathfrak{p}}^{+}(\mathrm{Ad}(\varphi))$, $V_{\mathfrak{p}}^{-}(\mathrm{Ad}(\varphi))$ or $V(\mathrm{Ad}(\varphi))$. Then we have the associated filtration of the full Selmer group:

$$\mathrm{Sel}_0(\mathrm{Ad}(\varphi)) \subset \mathrm{Sel}(\mathrm{Ad}(\varphi)) \subset \mathrm{Sel}_{-}(\mathrm{Ad}(\varphi)) \subset \mathrm{Sel}_{full}(\mathrm{Ad}(\varphi)).$$

Among these Selmer groups, $\mathrm{Sel}(\mathrm{Ad}(\varphi))$ associated to $V_{\mathfrak{p}}^{+}(\mathrm{Ad}(\varphi))$ is considered to be standard (generalizing the Selmer group in Example 2.3) and should be directly related to the normalized p-adic L-function described in [H96b] 4.3 (see the following example).

Example 2.7: We begin with a regular and pure motive M of rank n (with coefficients in $\mathbb{Q}$) defined over $E = \mathbb{Q}$. By the regularity, its Hodge numbers (p_i, q_i) for $i = 1, 2, \ldots, n$ satisfies $p_1 < p_2 < \cdots < p_n$. We consider the tensor product $M \otimes M^{\vee}$ and decompose it as $M \otimes M^{\vee} = \mathbf{1} \oplus \mathrm{Ad}(M)$. Then $\mathrm{Ad}(M)$ and $M \otimes M^{\vee}$ are motives of weight 0; so, we can write down its Hodge numbers in the following matrix form:

$$\begin{pmatrix} 0 & p_2 - p_1 & \cdots & p_n - p_1 \\ p_1 - p_2 & 0 & \cdots & p_n - p_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 - p_n & p_2 - p_n & \cdots & 0 \end{pmatrix}.$$

Thus $\mathrm{Ad}(M)$ is critical at 0 and 1 if complex conjugation acts by the scalar multiplication -1 on $H^{0,0}(\mathrm{Ad}(M))$, and the middle term of Hodge filtration $\mathcal{F}^{-}\mathrm{Ad}(M) = \mathcal{F}^{0}(\mathrm{Ad}(M) \subset H_{DR}(\mathrm{Ad}(M))$ corresponds to the upper triangular part of the above matrix, and $\mathcal{F}^{+}\mathrm{Ad}(M) = \mathcal{F}^{1}\mathrm{Ad}(M)$ corresponds to the upper nilpotent part of the matrix. When we twist $\mathrm{Ad}(M)$ by an Artin motive $M(\psi)$ of rank m with Galois representation ψ of $\Delta = \mathrm{Gal}(F/\mathbb{Q})$ for a totally real field F, the situation does not change. In other words, $\mathrm{Ad}(M) \otimes \psi$ and $\mathrm{Ad}(M)$ are critical at 0 and 1 at the same time, and $\mathcal{F}^{\pm}(\mathrm{Ad}(M) \otimes \psi) = (\mathcal{F}^{\pm}\mathrm{Ad}(M)) \otimes \psi$. We consider the p-adic Galois representation φ on the p-adic étale realization $H_p(M)$.

Since $\mathrm{Ad}(\varphi)$ is self dual, $L(1, \mathrm{Ad}(\varphi)\otimes\psi) \neq 0$ at least conjecturally, because $s = 1$ is the abscissa of convergence of $L(s, \mathrm{Ad}(\varphi)\otimes\psi)$. Since the conjectural functional equation is of the form $s \leftrightarrow 1 - s$, we should have $L(0, \mathrm{Ad}(\varphi)\otimes\psi) \neq 0$. We now suppose

(1) $\overline{\rho} = \varphi \bmod p$ is absolutely irreducible over F;

(2) φ restricted to D_p is isomorphic to an upper triangular representation with diagonal characters $\delta_1, \delta_2, \ldots, \delta_n$ from the top left corner;

(3) $\overline{\delta}_j = \delta_j \bmod p$ ($j = 1, 2, \ldots, n$) are all distinct on the inertia subgroup at p over F;

(4) $\mathbb{F}_p[S_F]$ and $\overline{\psi}$, which is ψ modulo the maximal ideal, are disjoint as Δ-modules,

where S_F is the set of primes over p in F and Δ acts on the space of formal linear combination $\mathbb{F}_p[S_F]$ through its action on S_F. The condition (2) asserts that the parabolic subgroup P_p is the standard Borel subgroup. This is necessary to have the unipotent radical $\mathfrak{n} \subset V(\mathrm{Ad}(\varphi))$ correspond to $\mathcal{F}^-(\mathrm{Ad}(M)(1)) = \mathcal{F}^+(\mathrm{Ad}(M))$ by the comparison isomorphism. Note that $n-1$ times the dimension of ψ-isotypic component of $\mathbb{F}_p[S_F]$ is equal to the number of linear p-Euler factor of $L(s, \mathrm{Ad}(\varphi)\otimes\psi)$ which vanishes at $s = 0$. Thus (4) implies no trivial zero occurs at $s = 0$. Thus the Selmer group $\mathrm{Sel}(\mathrm{Ad}(M)\otimes\psi)_{/\mathbb{Q}}$ should be finite. More generally if $\varphi\colon \mathfrak{G} \to \mathrm{GL}_n(\mathbb{J})$ specializes to the Galois representation associated to a motive M as above, the p-adic L-function should specialize to the order of $\mathrm{Sel}(\mathrm{Ad}(M)\otimes\psi)_{/\mathbb{Q}}$, and hence $\mathrm{Sel}^*(\mathrm{Ad}(\varphi)\otimes\psi)_{/\mathbb{Q}}$ has to be a torsion $\mathbb{J}$-module of finite type.

We now show that the motive $\mathrm{Ad}(M)$ is critical only when $n = 2$. Since complex conjugation c (Frobenius at ∞) reverses the Hodge filtration, we may assume that $\varphi(c)$ represents the longest element of the Weyl group of the maximal split torus of $\mathrm{GL}(n)$. Then the multiplicity of the eigenvalue -1 of its adjoint action on the Lie algebra $\mathfrak{t}$ of T is equal to $\left[\frac{n}{2}\right]$. This number has to be equal to $n - 1$ for $\mathrm{Ad}(M)$ to be critical. This happens only when $n = 2$.

As already remarked above, we cannot get a critical adjoint motive from $\mathrm{GL}(n)$ Galois representations if $n > 2$. To get something critical, we generalize a bit the definition of $\mathrm{Ad}(\varphi)$ and its Selmer group to general classical groups (when $p > 2$). We write $\mathrm{CNL}_{\mathcal{O}}$ for the category of complete noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. For any object $A \in \mathrm{CNL}_{\mathcal{O}}$, we write $\mathfrak{m}_A$ for its maximal ideal. We consider the following type of algebraic group G defined over $\mathcal{O}$: Let V be an $\mathcal{O}$-free module of rank n and $(\ ,\ )\colon V \times V \to \mathcal{O}$ be a symmetric or symplectic bilinear form with unit discriminant. For each object $A \in \mathrm{CNL}_{\mathcal{O}}$, we consider the induced bilinear form $(\, , \,)_A$: $V(A) \times V(A) \to A$ by $(\, , \,)$ for $V(A) = V \otimes_{\mathcal{O}} A$. Then

$$G(A) = \{g \in \mathrm{End}(V \otimes_{\mathcal{O}} A) \big| \, (gx, gy)_A = \nu(g)(x,y)_A\}$$

for all $x, y \in V \otimes_{\mathcal{O}} A$ with $\nu(g) \in A^{\times}$. We then define $\mathfrak{s}(A)$ to be the Lie-algebra of the derived group S of G over A. Then we define $\mathrm{Ad}_S(\varphi)(\sigma) = \varphi(\sigma)s\varphi(\sigma)^{-1}$ for $s \in \mathfrak{s}(\mathbb{J})$ and a representation φ: $\mathfrak{G} \to G(\mathbb{J})$. In this way, we get

$$\mathrm{Ad}_S(\varphi)\colon \mathfrak{G} \to \mathrm{GL}(\mathfrak{s}(\mathbb{J})). \tag{2.1}$$

For a character χ: $\mathfrak{G} \to \mathbb{J}^{\times}$, $\varphi \otimes \chi$ still has values in G, and $\mathrm{Ad}_S(\varphi) \cong \mathrm{Ad}_S(\varphi \otimes \chi)$. Let $\phi_{\det}$: $\mathfrak{G} \to \mathcal{O}^{\times}$ be the Teichmüller lift of $\det(\varphi \bmod \mathfrak{m}_{\mathbb{J}})$: $\mathfrak{G} \to \mathbb{F}^{\times}$. Similarly, we write ϕ_ν for the Teichmüller lift of $\nu(\varphi \bmod \mathfrak{m}_{\mathbb{J}})$. Then $\phi_{\det}^{-1}\det(\varphi)$ and $(\phi_\nu)^{-1}\nu(\varphi)$ are p-profinite characters. Thus if p is prime to $2n$, we have the unique n-th root Φ of $\phi^{-1}\det(\varphi)$ and the unique square root Φ' of ${\phi_\nu}^{-1}\nu(\varphi)$. Since $\det(\varphi)^2 = \nu(\varphi)^n$ and $\phi_{\det}^2 = (\phi_\nu)^n$, we see $\Phi^{2n} = (\Phi')^{2n}$; thus $\Phi = \Phi'$. We then put $\varphi_0 = \varphi \otimes \Phi^{-1}$. Then $\nu(\varphi) = \phi_\nu$ and $\det(\varphi) = \phi_{\det}$. Therefore we may assume, if p is prime to $2n$,

(Rt) $\det(\varphi)$ and $\nu(\varphi)$ have values in $\mathcal{O}^{\times}$;

(Ss) $\det(\varphi)$ and $\nu(\varphi)$ are of finite order, and their orders are prime to p.

Anyway, by extending scalar $\mathcal{O}$, we may assume (Rt) without assuming that p is prime to $2n$. On the other hand, to achieve the condition (Ss), we may need to assume that p is prime to $2n$.

Example 2.8: Let $M_{/\mathbb{Q}}$ be a rank n-motive with coefficients in $\mathbb{Q}$. We suppose that M is pure and regular. Thus $\mathbb{J} = \mathbb{Z}_p$ and $E = \mathbb{Q}$. We suppose that M has a polarization $\langle \, , \, \rangle_M$: $M \otimes M \to \mathbb{Q}(r)$ (see [DM] Section 4) for the Tate motive $\mathbb{Q}(r) = \mathbb{Q}(1)^{\otimes r}$. Then the Galois representation φ on the p-adic étale realization of M has values in the similitude group G of the symmetric or alternating form induced by $\langle \, , \, \rangle_M$. We suppose $V(\widetilde{\varphi}) \cong V(\varphi)(-r)$ for the contragredient $\widetilde{\varphi}$ under the polarization. By extending scalar to sufficiently large $\mathcal{O}$, we may assume that the group $G(\mathcal{O})$ is one of the types of groups we are studying. The polarization splits $M \otimes M^{\vee}$ into two pieces, symmetric part and alternating part with respect to the polarization: $M \otimes M^{\vee} = \mathrm{Sym}^2(M) \oplus \bigwedge^2 M$ (regarding $M \otimes M^{\vee} = \mathrm{End}(M)$). We define $\mathrm{Ad}(M)$ to be $\mathrm{Sym}^2(M)$ or $\bigwedge^2 M$ according as the parity of the polarization.

Now we assume that φ is nearly ordinary of Borel type. We may then assume, without losing much generality, that the Borel subgroup $B \subset G$ is associated to the Hodge filtration of the de Rham realization of the motive M under the comparison isomorphism. Since the complex conjugation c reverses the Hodge filtration, we may assume that the complex conjugation $\varphi(c)$ represents the longest element in the Weyl group of the split torus T of B. Then in this similitude group case, the adjoint action on the Lie algebra $\mathfrak{t}$ of T is scalar multiplication by -1 outside the center of G (basically by definition of G). Thus $\mathrm{Ad}(M)$ is always critical in this case (and hence $\mathrm{Ad}(M)(1)$ is also critical).

Hereafter we float the notation "G" and write G for the group which is the target group for Galois representations. Thus if we look into representation having values in $\mathrm{GL}_n(A)$ ($A \in \mathrm{CNL}_{\mathcal{O}}$) we write G for $\mathrm{GL}(n)_{/\mathcal{O}}$, otherwise G is the algebraic group introduced as above. We suppose

(Sp) G is split over $\mathcal{O}$.

(Pb) The stabilizer of the filtration $(\mathrm{fil}_{\mathfrak{p}})$ in $G(\mathbb{J})$ is equal to $P_{\mathfrak{p}}(\mathbb{J})$ for a parabolic subgroup $P_{\mathfrak{p}}$ of G.

Since G is split over $\mathcal{O}$, we have

(Sm) The group G and the center Z of G are both smooth over $\mathcal{O}$.

Since G is split over $\mathcal{O}$, the parabolic subgroup $P_{\mathfrak{p}}$ in (Pb) is conjugate to a standard one (defined over $\mathcal{O}$) in $G(\mathbb{J})$. By abusing the language, we sometimes call the conjugacy class in $G(A)$ of the standard parabolic subgroup the **class** of $P_{\mathfrak{p}}$ (over A).

Since $\mathfrak{s}(\mathbb{J})$ can be regarded as a subspace of $V(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))$, we can define

$$(\pm) \qquad V_{\mathfrak{p}}^{\pm}(\mathrm{Ad}_S(\varphi)) = V(\mathrm{Ad}_S(\varphi)) \bigcap V_{\mathfrak{p}}^{\pm}(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi)).$$

Then we define $\mathrm{Sel}(\mathrm{Ad}_S(\varphi))$ and $\mathrm{Sel}_-(\mathrm{Ad}_S(\varphi))$ with respect to $V^+(\mathrm{Ad}_S(\varphi))$ and $V^-(\mathrm{Ad}_S(\varphi))$, respectively.

2.4. Universal deformation rings. We suppose that $\mathbb{J} \in \mathrm{CNL}_{\mathcal{O}}$. We put $\overline{\rho} = \varphi \bmod \mathfrak{m}_{\mathbb{J}}\colon \mathfrak{G} \to G(\mathbb{F})$. We write $\overline{\delta}_{j,\mathfrak{p}}$ for $\delta_{\overline{\rho},j,\mathfrak{p}}$. Let $\mathcal{P}_{\mathfrak{p}}$ be the Lie algebra of $P_{\mathfrak{p}}$ in (Pb). Then for the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{g}/\mathcal{P}_{\mathfrak{p}}$ has a natural filtration induced by $(\mathrm{fil}_{\mathfrak{p}})$, which is stable under the adjoint action of $D_{\mathfrak{p}}$. Let $gr(\mathfrak{g}/\mathcal{P}_{\mathfrak{p}})$ be the graded module under this filtration. When $G = \mathrm{GL}(n)$, the filtration on $\mathfrak{gl}$ is the double filtration induced by $\mathfrak{gl}(\mathbb{J}) = V(\varphi) \otimes V(\widetilde{\varphi})$ for the contragredient $\widetilde{\varphi}$ of φ. In particular,

$$\mathrm{gr}((\mathfrak{gl}/\mathcal{P}_{\mathfrak{p}})(\mathbb{J})) \cong \bigoplus_{i>j} V(\delta_{j,\mathfrak{p}}) \otimes V(\widetilde{\delta}_{i,\mathfrak{p}}) \cong \bigoplus_{i>j} \mathrm{Hom}_{\mathbb{J}}(\delta_{j,\mathfrak{p}}, \delta_{i,\mathfrak{p}})$$

as $D_{\mathfrak{p}}$-modules. When $G \neq \mathrm{GL}(n)$, $\mathrm{gr}((\mathfrak{g}/\mathcal{P}_{\mathfrak{p}})(\mathbb{J}))$ can be **identified** with a submodule of the above. In other words, writing the Levi-component $M_{\mathfrak{p}}$ of $P_{\mathfrak{p}}$

as $M_1 \times M_2 \times \cdots \times M_{m_\mathfrak{p}}$ for the split rank $m_\mathfrak{p}$ of the center of $M_\mathfrak{p}$ so that the derived group of M_i is either simple or trivial, we have

$$\mathrm{gr}((\mathfrak{g}/\mathcal{P}_\mathfrak{p})(\mathbb{J})) \subset \bigoplus_{1\le j<i\le m_\mathfrak{p}} V(\delta_{j,\mathfrak{p}}) \otimes V(\widetilde{\delta}_{i,\mathfrak{p}}) \cong \bigoplus_{1\le j<i\le m_\mathfrak{p}} \mathrm{Hom}_{\mathbb{J}}(\delta_{j,\mathfrak{p}}, \delta_{i,\mathfrak{p}})$$

for the projection $\delta_{i,\mathfrak{p}}$ of $\varphi|_{D_\mathfrak{p}}$ to $M_i(\mathbb{J})$.

Let $D_\mathfrak{p}$ be the decomposition group at $\mathfrak{p} \in S_L$ in $\mathfrak{H}_L$, and we write $I_\mathfrak{p}$ for the inertia subgroup of $D_\mathfrak{p}$. We consider the following four conditions (cf. [Til] Chapter 6):

(AI_L) $\overline{\rho}_L$ is absolutely irreducible as a representation of $\mathfrak{H}_L$ into $\mathrm{GL}_n(\mathbb{F})$;

(Z_L) The centralizer of each deformation $\rho\colon \mathfrak{H}_L \to \mathrm{GL}_n(A)$ ($A \in \mathrm{CNL}_\mathcal{O}$) of $\overline{\rho}_L$ is made of scalar matrices in $\mathrm{GL}_n(A)$;

($\mathrm{Z}_{p,L}$) The centralizer of each deformation $\delta\colon I_\mathfrak{p} \to \mathrm{GL}_m(A)$ ($A \in \mathrm{CNL}_\mathcal{O}$) of $\overline{\delta}_{i,\mathfrak{p}}$ is made of scalar matrices in $\mathrm{GL}_m(A)$ for all $\mathfrak{p}|p$ in L and i;

(Reg_L) $$H^0(D_\mathfrak{p}, (\mathfrak{g}/\mathcal{P}_\mathfrak{p})(\mathbb{F})) = 0 \text{ for all } \mathfrak{p} \in S_L.$$

The last condition (Reg_L) follows from the following condition:

(RG_L) $$\mathrm{Hom}_{D_\mathfrak{p}}(V(\overline{\delta}_{i,\mathfrak{p}}), V(\overline{\delta}_{j,\mathfrak{p}})) = 0 \quad \text{for all } \mathfrak{p} \in S_L \text{ and } 1 \le j < i \le m_\mathfrak{p}.$$

LEMMA 2.2: *Suppose that L/M with $E \subset M \subset L \subset F^{(p,\infty)}$ is a finite Galois extension with p-power $[L : M]$. Then*

1. *(AI_M) is equivalent to (AI_L) if $p \nmid n$;*
2. *If $p \nmid n$, (Z_M) $\iff$ (Z_L) , and if $p \nmid \prod_{j,\mathfrak{p}} \dim \overline{\delta}_{j,\mathfrak{p}}$, ($\mathrm{Z}_{p,F}$) $\iff$ ($\mathrm{Z}_{p,L}$);*
3. *(RG_M) (resp. (Reg_M)) is equivalent to (RG_L) (resp. (Reg_L)).*

Proof: We first prove (1). Since $\mathrm{Gal}(L/M)$ is nilpotent,. we may assume that L/M is cyclic. Suppose that $\overline{\rho}_L$ is reducible, and write $\overline{\xi}$ for one of absolutely irreducible subquotients. Let $H \subset \mathrm{Gal}(L/M)$ be the stabilizer of $\overline{\xi}$. Then writing $L' = L^H$, since L/L' is p-power cyclic, $\overline{\xi}$ extends uniquely to a representation $\overline{\xi}_{L'}$ of $\mathfrak{H}'_L = \mathrm{Gal}(F^{(p,\infty)}/L')$ (because other extensions are of the form $\overline{\xi}_{L'} \otimes \chi$ for a character χ: $\mathrm{Gal}(L/L') \to \mathbb{F}^\times$, which is trivial). Thus by the Frobenius reciprocity law, $\overline{\rho}_{L'}$ becomes already reducible, containing $\overline{\xi}_{L'}$. Thus we may assume that $H = \{1\}$ and $L = L'$. Then by the absolute irreducibility of $\overline{\xi}$, $\mathrm{Hom}_{\mathfrak{H}_L}(\overline{\xi}, \overline{\xi}^\sigma) = 0$ for all non-trivial $\sigma \in \mathrm{Gal}(L/M)$. Therefore by Mackey's theorem, $\mathrm{Ind}_L^M \overline{\xi}$ is irreducible, and hence $\overline{\rho} \cong \mathrm{Ind}_L^M \overline{\xi}$. In particular, we have $n = \dim \overline{\rho} = [L : M] \dim \overline{\xi}$, and hence $[L : M] = 1$, because $p \nmid n$. This shows the first assertion.

We now prove the first half of (2). Let $\rho\colon \mathfrak{H}_M \to \mathrm{GL}_n(A)$ be a deformation of $\overline{\rho}_M$ for an Artinian local $\mathcal{O}$-algebra A. By the assumption $p \nmid n$, $\mathrm{End}_A(V(\rho)) = V \oplus A$ as $\mathfrak{H}_M$-modules for $V = V(\mathrm{Ad}_{\mathrm{SL}(n)}(\rho))$. Write

$$Z(\rho) = \{x \in \mathrm{GL}_n(A) |\, x\rho x^{-1} = \rho\}.$$

Then $Z(\rho) = A^\times \iff H^0(\mathfrak{H}_M, V) = 0$. The action of $\mathfrak{H}_M$ on $W = H^0(\mathfrak{H}_L, V)$ factors through $\Delta = \mathrm{Gal}(L/M)$. By definition, $H^0(\Delta, W) = H^0(\mathfrak{H}_M, V) = 0$. Let W^* be the Pontryagin dual of W. Then $H^0(\Delta, W)$ is the Pontryagin dual of $W/\mathfrak{a}W$ for the augmentation ideal $\mathfrak{a}$ of $A[\Delta]$. Since Δ is a finite p-group, $\mathfrak{a}$ is nilpotent. Thus by Nakayama's lemma, $W = 0 \iff W^* = 0 \iff W^*/\mathfrak{a}W^* = 0 \iff H^0(\mathfrak{H}_M, V) = 0$. This shows the result. The same proof applies to $(\mathrm{Z}_{p,L})$.

We now prove (3). We look at

$$V = \mathrm{Hom}_{\mathbb{F}}(V(\overline{\delta}_{i,\mathfrak{p}}), V(\overline{\delta}_{j,\mathfrak{p}})) \quad \text{and} \quad W = H^0(\mathfrak{H}_L, V).$$

Then (RG_M) is equivalent to $H^0(D_\mathfrak{p}, V) = 0$ for all $j < i$. Since $\Delta_\mathfrak{p} \subset \Delta$ is a p-group, by the above argument,

$$H^0(D_\mathfrak{p}, V) = H^0(\Delta_\mathfrak{p}, W) = 0 \iff W = 0,$$

which shows the assertion for (RG). The same argument applied to $V = (\mathfrak{g}/\mathcal{P}_\mathfrak{p})(\mathbb{F})$ yields the equivalence for (Reg). ∎

Two deformations ρ and ρ' of $\overline{\rho}$ (with values in $G(A)$) are strictly equivalent if $\rho(g) = x\rho'(g)x^{-1}$ for $x \in \widehat{G}(A)$ for the formal group $\widehat{G}$, that is,

$$\widehat{G}(A) = \{x \in G(A) |\, x \equiv 1 \bmod \mathfrak{m}_A\}.$$

We write $\rho \approx \rho'$ if they are strictly equivalent. A deformation ρ is called **nearly ordinary** of type $\mathcal{F} = \{P_\mathfrak{p}\}$ if we have the following filtration of $D_\mathfrak{p}$-modules for all $\mathfrak{p} \in S$ whose stabilizer is in the class of $P_\mathfrak{p}$ over A:

$$(\mathrm{fil}_\mathfrak{p}) \qquad 0 \subset V(\rho)_{1,\mathfrak{p}} \subset V(\rho)_{2,\mathfrak{p}} \subset \cdots \subset V(\rho)_{m_\mathfrak{p},\mathfrak{p}} = V(\rho),$$

where $V(\rho)/V(\rho)_{j,\mathfrak{p}}$ is a A-free module and $\delta_{\rho,j,\mathfrak{p}} \bmod \mathfrak{m}_A \cong \overline{\delta}_{j,\mathfrak{p}}$ for all j.

We assume that (Rt) and write $\phi_{\det} = \det(\varphi)$ and $\phi_\nu = \nu(\varphi)$. We put ϕ for the pair $(\phi_{\det}, \phi_\nu)$ if $G \neq \mathrm{GL}(n)$ and $\phi = \phi_{\det}$ if $G = \mathrm{GL}(n)$. Since these characters have values in $\mathcal{O}^\times$, we may regard them as characters having values in any object $A \in \mathrm{CNL}_\mathcal{O}$ by composing the structure homomorphism $\mathcal{O} \to A$. Under (Sm), (Z_L) and (Reg_L), the functor $\Phi_L^\phi = \Phi_{G,L}^\phi$, associating to $A \in \mathrm{CNL}_\mathcal{O}$

strict equivalence classes of type $\mathcal{F}$ deformations ρ: $\mathfrak{H}_L \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ with $(\det(\rho), \nu(\rho)) = \phi$, is representable as shown by Mazur, Boston and Tilouine (cf. [Til] Chapter 6), where $\phi_{\det}$ and ϕ_ν are regarded as having values in A via the structure homomorphism: $\mathcal{O} \to A$. Thus there exists a unique couple $(R, \varrho) = (R^{\phi}_{G,L}, \varrho^{\phi}_{G,L})$ made of $R \in \mathrm{CNL}_{\mathcal{O}}$ and a continuous deformation ϱ: $\mathfrak{H}_L \to G(R)$ of $\overline{\rho}$ such that for each nearly ordinary type $\mathcal{F}$ deformation ρ: $\mathfrak{H}_L \to G(A) \in \Phi^{\phi}_{G,L}(A)$ of $\overline{\rho}$, there exists a unique $\mathcal{O}$-algebra homomorphism ι_ρ: $R \to A$ such that ρ is strictly equivalent to $\iota_\rho \circ \varrho$. In particular, we have a unique $\mathcal{O}$-algebra homomorphism π: $R \to \mathbb{J}$ such that $\pi \circ \varrho \approx \varphi$.

Write $\overline{\xi} = \overline{\delta}_{j,\mathfrak{p}}$. Under the assumption $(\mathrm{Z}_{p,L})$, the deformation functor for $H = I_{\mathfrak{p}}$ or $D_{\mathfrak{p}}$:

$$\Phi^H_{\mathfrak{p},\overline{\xi}}(A) = \left\{\xi\colon H \to \mathrm{GL}_{\dim(\overline{\xi})}(A) \middle| \, \xi \bmod \mathfrak{m}_A = \overline{\xi}\right\}/\approx$$

is representable over $\mathrm{CNL}_{\mathcal{O}}$. We write $(R^H_{\mathfrak{p},\overline{\xi}}, \varrho^H_{\mathfrak{p},\overline{\xi}})$ for the universal couple. For the universal deformation $\varrho \in \Phi^{\phi}(R)$ of type $\mathcal{F}$, we have $\delta_{\varrho,j,\mathfrak{p}}$: $H \to \mathrm{GL}_{\dim \overline{\delta}_{j,\mathfrak{p}}}(R)$, which is a deformation of $\overline{\delta}_{j,\mathfrak{p}}$ over H. Thus we have a canonical $\mathcal{O}$-algebra homomorphism $i_{j,\mathfrak{p}}$: $R^H_{\mathfrak{p},\overline{\delta}_{j,\mathfrak{p}}} \to R$ inducing $\delta_{\varrho,j,\mathfrak{p}}$ from the corresponding universal representation of H. For $H = I$ and D, we write $R^H_{\mathrm{GL}(n),F}$ for $\widehat{\bigotimes}_{j,\mathfrak{p}}(R^{H_{\mathfrak{p}}}_{\mathfrak{p},\overline{\delta}_{j,\mathfrak{p}}})$ and ι_H: $R_H \to R$ for the tensor product of these morphisms. Again by definition, $R^D_{\mathrm{GL}(n),L}$ is naturally an $R^I_{\mathrm{GL}(n),L}$-algebra.

Let $M_{\mathfrak{p}}$ be the (standard) Levi subgroup of the standard parabolic subgroup in G conjugate to $P_{\mathfrak{p}}$. Then we can regard the representation on $\mathrm{gr}(V(\overline{\rho}_{\mathfrak{p}})) = \bigoplus_j V(\overline{\delta}_{j,\mathfrak{p}})$ as a representation $\overline{\delta}_{\mathfrak{p}}$ of $D_{\mathfrak{p}}$ having values in $M(\mathbb{F})$. Then we can think of the deformation functor for $H = D_{\mathfrak{p}}$ and $I_{\mathfrak{p}}$:

$$\Psi^H_{\mathfrak{p}}(A) = \left\{\rho\colon H \to M_{\mathfrak{p}}(A) \middle| \, \rho \equiv \overline{\delta}_{\mathfrak{p}} \bmod \mathfrak{m}_A\right\}/\widehat{M}_{\mathfrak{p}}(A),$$

where $\widehat{M}_{\mathfrak{p}}(A) = \{x \in M_{\mathfrak{p}}(A) |\, x \equiv 1 \bmod \mathfrak{m}_A\}$. Under $(\mathrm{Z}_{p,L})$ for individual $\overline{\delta}_{j,\mathfrak{p}}$ for each j, this functor is representable, giving rise to the universal ring $R^H_{\mathfrak{p},G}$. We put

$$R^D_{G,L} = \widehat{\bigotimes}_{\mathfrak{p}\in S_F} R^{D_{\mathfrak{p}}}_{G,L} \quad \text{and} \quad R^I_{G,L} = \widehat{\bigotimes}_{\mathfrak{p}\in S_F} R^{I_{\mathfrak{p}}}_{G,L}. \tag{2.2}$$

The global universal deformation ring $R^{\phi}_{G,L}$ is an algebra over the local one $R^H_{G,L}$ for $H = I$ and D, and the algebra structure of $R^{\phi}_{G,L}$ over $R^H_{\mathrm{GL}(n),L}$ for $H = I$ and D factors through $R^H_{G,L}$.

2.5. KÄHLER DIFFERENTIALS ON UNIVERSAL DEFORMATION RINGS. We recall Mazur's argument (cf. [MT]) to relate 1-differentials on $\mathrm{Spec}(R)$ ($R = R^{\phi}_{G,L}$) with the Selmer group $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/L}$.

For any R-module X of finite type, we write $R[X]$ for the R-algebra with square zero ideal X. Thus $R[X] = R \oplus X$ with $(r \oplus x)(r' \oplus x') = rr' \oplus (rx' + r'x)$. It is easy to see that $R[X] \in \mathrm{CNL}_{\mathcal{O}}$. We consider the $\mathcal{O}$-algebra homomorphism $\xi: R \to R[X]$ with $\xi \bmod X = \mathrm{id}$. Then we can write $\xi(r) = r \oplus d_\xi(r)$ with $d_\xi(r) \in X$. By the above definition of the product, we get $d_\xi(rr') = rd_\xi(r') + r'd_\xi(r)$ and $d_\xi(\mathcal{O}) = 0$. Thus d_ξ is a derivation, i.e., $d_\xi \in \mathrm{Der}_{\mathcal{O}}(R, X)$. For any derivation $d: R \to X$ over $\mathcal{O}$, $r \mapsto r \oplus d(r)$ is obviously an $\mathcal{O}$-algebra homomorphism, and we get

$$
\begin{aligned}
(2.3) \qquad \{\, \rho \in \Phi^{\phi}_{G,L}(R[X]) \mid \rho \bmod X = \varrho \,\} / \approx_X \\
&\cong \left\{ \rho \in \Phi^{\phi}_{G,L}(R[X]) \mid \rho \bmod X \approx \varrho \right\} / \approx \\
&\cong \left\{ \xi \in \mathrm{Hom}_{\mathcal{O}\text{-}alg}(R, R[X]) \mid \xi \bmod X = \mathrm{id} \right\} \\
&\cong \mathrm{Der}_{\mathcal{O}}(R, X) \cong \mathrm{Hom}_R(\Omega_{R/\mathcal{O}}, X),
\end{aligned}
$$

where "$\approx_X$" is conjugation under $1 \oplus M_n(X) \cap G(R[X])$. Here and hereafter $\Omega_{A/B}$ for a B-algebra A ($A, B \in \mathrm{CNL}_{\mathcal{O}}$) indicates the module of continuous 1-differentials with respect to the profinite topology.

Let ρ be the deformation in the left-hand side of (2.3). Then we may write $\rho(\sigma) = \varrho(\sigma) \oplus u'_\rho(\sigma)$. We see

$$\varrho(\sigma\tau) \oplus u'_\rho(\sigma\tau) = (\varrho(\sigma) \oplus u'_\rho(\sigma))(\varrho(\tau) \oplus u'_\rho(\tau)) = \varrho(\sigma\tau) \oplus (\varrho(\sigma)u'_\rho(\tau) + u'_\rho(\sigma)\varrho(\tau)),$$

and we have

$$u'_\rho(\sigma\tau) = \varrho(\sigma)u'_\rho(\tau) + u'_\rho(\sigma)\varrho(\tau).$$

Define $u_\rho(\sigma) = u'_\rho(\sigma)\varrho(\sigma)^{-1}$.

On the other hand, $x(\sigma) = \rho(\sigma)\varrho(\sigma)^{-1}$ has values in $S(R[X])$, and $x = 1 \oplus u \mapsto u = x - 1$ is an isomorphism from the multiplicative group of the kernel of the reduction map $S(R[X]) \twoheadrightarrow S(R)$: $\{x \in S(R[X]) \mid x \equiv 1 \bmod X\}$ onto the additive group $\mathrm{Ad}_S(X) = \mathfrak{s}(R) \otimes_R X = V(\mathrm{Ad}_S(\varrho)) \otimes_R X$. Thus we may regard u as having values in $\mathrm{Ad}_S(X) = V(\mathrm{Ad}_S(\varrho)) \otimes_R X$.

We also have

$$
\begin{aligned}
(2.4) \qquad u_\rho(\sigma\tau) &= u'_\rho(\sigma\tau)\varrho(\sigma\tau)^{-1} \\
&= \varrho(\sigma)u'_\rho(\tau)\varrho(\sigma\tau)^{-1} + u'_\rho(\sigma)\varrho(\tau)\varrho(\sigma\tau)^{-1} \\
&= \mathrm{Ad}_S(\varrho)(\sigma)u_\rho(\tau) + u_\rho(\sigma).
\end{aligned}
$$

Hence $u_\rho\colon \mathfrak{H}_L \to \mathrm{Ad}_S(X)$ is a 1-cocycle. It is easy to see the injectivity of the map:

$$\left\{\rho \in \Phi^{\phi}_{G,L}(R[X]) \middle|\, \rho \bmod X \approx \varrho\right\} / \approx_X \hookrightarrow H^1(\mathfrak{H}_L, \mathrm{Ad}_S(X))$$

given by $\rho \mapsto [u_\rho]$. We put $V^{\pm}_{\mathfrak{p}}(\mathrm{Ad}_S(X)) = V^{\pm}_{\mathfrak{p}}(\mathrm{Ad}_S(\varrho)) \otimes_R X$. Then we see that

$$(2.5)\quad u_\rho(I_{\mathfrak{p}}) \subset V^{+}_{\mathfrak{p}}(\mathrm{Ad}_S(X)) \iff u'_\rho(I_{\mathfrak{p}}) \subset V^{+}_{\mathfrak{p}}(\mathrm{Ad}_S(X)) \iff d_\xi(R^{I}_{\overline{\delta}_{\mathfrak{p}}}) = 0$$

if $\xi \in \mathrm{Hom}_{\mathcal{O}-alg}(R, R[X])$ induces ρ.

Note that $\mathbb{J}^* = \bigcup_X X$ for R-modules X of finite type, which shows $R[\mathbb{J}^*] = \bigcup_X R[X]$. From this, any deformation (continuous in an appropriate sense) having values in $G(R[\mathbb{J}^*])$ gives rise to a continuous cocycle (see [HT] Chapter 2 for details about continuity). In this way, we get

$$(2.6)\qquad (\Omega_{R/A} \otimes_R \mathbb{J})^* \cong \mathrm{Hom}_R(\Omega_{R/\mathcal{O}}, \mathbb{J}^*) \hookrightarrow H^1(\mathfrak{H}_L, V(\mathrm{Ad}_S(\varphi))^*)$$

if R is an A-algebra for an object A of $\mathrm{CNL}_{\mathcal{O}}$. Basically by definition, ρ is nearly p-ordinary if and only if u_ρ restricted to $I_{\mathfrak{p}}$ has values in $V^-(\mathrm{Ad}_S(\varphi))^*$.

We can argue in the same way as above, replacing the inertia groups by the decomposition groups, and we get the corresponding results on the strict Selmer groups. Thus we get from this and (2.5) the following fact:

THEOREM 2.3: *Let $G = \mathrm{GL}(n)$ or the group introduced in 2.4. Let S be the derived subgroup of G. Suppose (Z_L), (Reg_L) and $(\mathrm{Z}_{p,L})$ for $\overline{\delta}_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. Then*

$$\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/L} \cong \Omega_{R^{\phi}_{G,L}/R^{I}_{G,L}} \otimes_{R^{\phi}_{G,L}} \mathbb{J}, \quad \textit{and}$$
$$\mathrm{Sel}^*_-(\mathrm{Ad}_S(\varphi))_{/L} \cong \Omega_{R^{\phi}_{G,L}/\mathcal{O}} \otimes_{R^{\phi}_{G,L}} \mathbb{J}, \quad \mathrm{Sel}^*_{st}(\mathrm{Ad}_S(\varphi))_{/L} \cong \Omega_{R^{\phi}_{G,L}/R^{D}_{G,L}} \otimes_{R^{\phi}_{G,L}} \mathbb{J}$$

as $\mathbb{J}$-modules. Moreover, we have the following exact sequence:

$$\Omega_{R^{D}_{G,L}/R^{I}_{G,L}} \otimes_{R^{D}_{G,L}} \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/L} \to \mathrm{Sel}^*_{st}(\mathrm{Ad}_S(\varphi))_{/L} \to 0.$$

2.6. TWISTED SELMER GROUPS. Let $\varphi\colon \mathrm{Gal}(F^{(p,\infty)}/E) \to \mathrm{GL}_n(\mathbb{J})$ be a representation. Let $\overline{\rho} = \varphi \bmod \mathfrak{m}_{\mathbb{J}}$. Suppose that $\overline{\rho}_E$ and φ_E are nearly ordinary of type $\mathcal{F} = \{(\mathrm{fil}_{\mathfrak{p}})\}_{\mathfrak{p}}$. Fix a Selmer datum $\mathcal{S} = \{V^-_{\mathfrak{p}}\}$ for φ_E. In this section, we write $\mathrm{Sel}(\varphi)_{/L}$ for $\mathrm{Sel}_{\mathcal{S}}(\varphi)_{/L}$. We introduce the Selmer datum for $\mathrm{Ind}^E_F\,\varphi_F$ and $\varphi \otimes \psi$ (for an Artin representation ψ of $\Delta = \mathrm{Gal}(F/E)$) induced from $\mathcal{S}$ and study the relation between $\mathrm{Sel}(\varphi_E \otimes \psi)$, $\mathrm{Sel}(\mathrm{Ind}^E_F\,\varphi_F)$ and $\mathrm{Sel}(\varphi_F)$.

Since $[F : E]$ is prime to p, if $\mathbb{F}$ is sufficiently large, we can decompose $\mathrm{Ind}^E_F\,\mathbb{F}$ (for the trivial $\mathfrak{H}$-module $\mathbb{F}$) into a sum of absolutely irreducible representations

$\overline{\psi}: \Delta = \mathrm{Gal}(F/E) \to \mathrm{GL}_{m(\psi)}(\mathbb{F})$:

$$\mathrm{Ind}_F^E \mathbb{F} = \bigoplus_{\overline{\psi}} m(\psi)\overline{\psi}$$

with multiplicity $m(\psi) = \dim(\psi)$. Then this decomposition lifts to a unique decomposition of $\mathcal{O}[\Delta]$-modules: $\mathcal{O}[\Delta] = \bigoplus_{\overline{\psi}} m(\psi)\psi$ for a representation $\psi: \Delta \to \mathrm{GL}_{m(\psi)}(\mathcal{O})$ such that $\psi \bmod \mathfrak{m}_{\mathcal{O}} = \overline{\psi}$. The ψ-isotypic component $\mathcal{O}[\Delta][\psi]$ of $\mathcal{O}[\Delta]$ is an $\mathcal{O}$-algebra direct summand of $\mathcal{O}[\Delta]$ because of $p \nmid |\Delta|$. We write 1_ψ for the central idenpotent of $\mathcal{O}[\Delta][\psi]$. Then for any $\mathcal{O}[\Delta]$-module X, we write $X[\psi] = 1_\psi X$ and call it the ψ-isotypic component of X.

Let $\varphi_F = \varphi_E|_{\mathfrak{H}}$. Let L be an extension of E linearly disjoint from F over E. We put $M = LF$. The natural action of Δ on $H^1(\mathfrak{G}, V(\varphi_F)^*)$ induces the action of Δ on $\mathrm{Sel}_S(\varphi_F)_{/M}$. We then write $\mathrm{Sel}(\varphi_F)_{/M}[\psi]$ for the ψ-isotypic component of $\mathrm{Sel}(\varphi_F)_{/M}$. We like to give a Galois-cohomological definition of $\mathrm{Sel}(\varphi_E \otimes_{\mathcal{O}} \psi)_{/L}$ so that

$$\text{(TW)} \qquad \mathrm{Sel}(\varphi_F)_{/M}[\psi] \cong (\mathrm{Sel}(\varphi_E \otimes_{\mathcal{O}} \psi)_{/L})^{m(\psi)}$$

as $\mathbb{J}[[\mathrm{Gal}(L/E)]]$-modules if L/E is a Galois extension. By linear-disjointness of L and F over E, we have res : $\mathrm{Gal}(M/L) \cong \Delta$ by the restriction map. Let Σ be the set of primes ramifying in $F^{(p,\infty)}/E$, and let $E^{(\Sigma,\infty)}/E$ be the maximal extension unramified outside $\Sigma \cup \{\infty\}$. Let $\mathcal{G}_X = \mathrm{Gal}(E^{(\Sigma,\infty)}/X)$ for a subfield X of $E^{(\Sigma,\infty)}$. Then we define two $\mathcal{G}_E$-modules in the following way:

$$\mathrm{Ind}_F^E(V(\varphi_F)) = \mathcal{O}[\mathcal{G}_E] \otimes_{\mathcal{O}[\mathcal{G}_F]} V(\varphi_F) \quad \text{and} \quad \mathcal{O}[\Delta] \otimes_{\mathcal{O}} V(\varphi_E).$$

Here we regard $\mathcal{O}[\mathcal{G}_E] \otimes_{\mathcal{O}[\mathcal{G}_F]} V(\varphi_F)$ (resp. $\mathcal{O}[\Delta] \otimes_{\mathcal{O}} V(\varphi_E)$) as a left $\mathcal{G}_E$-module by $\sigma(\tau \otimes v) = \sigma\tau \otimes v$ (resp. $\sigma(\overline{\tau} \otimes v) = \overline{\sigma\tau} \otimes \sigma v$), where $\overline{\sigma}$ is the restriction of σ to F. We claim, as $\mathcal{O}[\mathcal{G}_E]$-modules,

$$\text{(2.7)} \qquad \iota : \mathrm{Ind}_F^E(V(\varphi_F)) \cong \mathcal{O}[\Delta] \otimes_{\mathcal{O}} V(\varphi_E).$$

The isomorphism ι is given by

$$\mathcal{O}[\mathcal{G}_E] \otimes_{\mathcal{O}[\mathcal{G}_F]} V(\varphi_F) \ni \sigma \otimes v \mapsto \overline{\sigma} \otimes \sigma v \in \mathcal{O}[\Delta] \otimes_{\mathcal{O}} V(\varphi_E),$$

where $\sigma \mapsto \overline{\sigma}$ indicates the projection of $\mathcal{G}_E$ onto Δ.

(1) By Shapiro's lemma, we have $H^1(\mathcal{G}_L, \mathrm{Ind}_F^E V(\varphi_F)^*) \cong H^1(\mathcal{G}_M, V(\varphi_F)^*)$.

(2) On the other hand, from (2.7), we have

$$H^1(\mathcal{G}_L, \mathrm{Ind}_F^E V(\varphi_F)^*) \cong \bigoplus_{\psi} H^1(\mathcal{G}_L, V(\varphi_E \otimes \psi)^*)^{m(\psi)}.$$

We can let Δ act on $\mathcal{O}[\Delta]\otimes V(\varphi_E)$ by $\delta(a\otimes v) = a\delta^{-1}\otimes v$. This action commutes with the action of $\mathcal{G}_E$ on $\mathrm{Ind}_F^E\,\varphi_F$. Since ψ is self-$\mathcal{O}$-dual ($p\nmid|\Delta|$),

$$(\mathcal{O}[\Delta]\otimes V(\varphi_E))[\psi] = (\mathcal{O}[\Delta][\psi])\otimes V(\varphi_E).$$

Then, combining the two identities (1) and (2), we see that, for $V = V(\varphi_E\otimes\psi)$,

$$\iota_* : H^1(\mathcal{G}_L, V^*)^{m(\psi)} \longrightarrow H^0(\Delta, H^1(\mathcal{G}_M, V^*))^{m(\psi)} \cong H^1(\mathcal{G}_M, V(\varphi_E)^*)[\psi] \tag{2.8}$$

is an isomorphism of $\mathcal{O}[\mathrm{Gal}(L/E)]$-modules, identifying Δ with $\mathrm{Gal}(M/L)$.

There is another way of showing the isomorphism (2.8): Writing $V = V(\varphi_E\otimes\psi)$, we have from the inflation-restriction sequence:

$$\begin{aligned}0 \to H^1(\Delta, H^0(\mathcal{G}_M, V^*)) \to H^1(\mathcal{G}_L, V^*) &\xrightarrow{\iota} H^0(\Delta, H^1(\mathcal{G}_M, V^*))\\ &\cong \mathrm{Hom}_\Delta(V(\psi), H^1(\mathcal{G}_M, V(\varphi_E)^*)) \to H^2(\Delta, H^0(\mathcal{G}_M, V^*)).\end{aligned}$$

Since $d = [F:E]$ is prime to p, $H^q(\Delta, H^0(\mathcal{G}_M, V(\varphi_E\otimes\psi)^*)) = 0$ for all $q > 0$, and hence we get the isomorphism (2.8), because

$$\mathrm{Hom}_\Delta(V(\psi), H^1(\mathcal{G}_M, V(\varphi_E)^*)^{m(\psi)} \cong H^1(\mathcal{G}_M, V(\varphi_E)^*)[\psi].$$

We can give another definition of $\mathrm{Sel}(\varphi_F)_{/M}$ equivalent to the original one, using $\mathcal{G}_M$ in place of $\mathfrak{H} = \mathrm{Gal}(F^{(p,\infty)}/F)$:

$$\begin{aligned}\mathrm{Sel}(\varphi_F)_{/M} =& \mathrm{Ker}(H^1(\mathcal{G}_M, V(\varphi_F)^*)\\ &\xrightarrow{r} \prod_{\mathfrak{Q}\nmid p} H^1(I_{\mathfrak{Q}}, V(\varphi_F)^*) \times \prod_{\mathfrak{P}|p} H^1(I_{\mathfrak{P}}, V(\varphi_F)^*/V_{\mathfrak{P}}^-(\varphi_F)^*)),\end{aligned} \tag{2.10}$$

where capital gothic characters indicate prime ideals of M and r is the product of the restriction maps for $\mathfrak{Q}\nmid p$ and those composed with the projections:

$$V(\varphi_F)^* \to \frac{V(\varphi_F)^*}{V_{\mathfrak{P}}^-(\varphi_F)^*}$$

for $\mathfrak{P}|p$.

There is a local version of the restriction map ι: First we fix a prime ideal $\mathfrak{P}$ of M dividing a prime ideal $\mathfrak{p}|p$ in L such that $V_{\mathfrak{P}}^-(\varphi_F) = V_{\mathfrak{p}}^-(\varphi_E)$. All other primes over $\mathfrak{p}$ can be written as $\sigma(\mathfrak{P})$ for $\sigma \in \mathcal{G}_M\backslash\mathcal{G}_L/D_{\mathfrak{p}}$. Thus $D_{\sigma(\mathfrak{P})} = \sigma D_{\mathfrak{P}}\sigma^{-1}$ and $V_{\sigma(\mathfrak{P})}^-(\varphi_F) = \sigma(V_{\mathfrak{P}}^-(\varphi_F))$. Note that $\mathcal{G}_M\sigma D_{\mathfrak{p}} = \sigma\mathcal{G}_M D_{\mathfrak{p}} = \sigma D_{\mathfrak{p}}\mathcal{G}_M$, and hence $\mathcal{O}[\mathcal{G}_M\sigma D_{\mathfrak{p}}]$ is a right and left $\mathcal{G}_M$-module. Note that $\{\mathfrak{P}'|\mathfrak{p}\} = \{\sigma(\mathfrak{P})\} \cong \{\mathcal{G}_M\backslash\mathcal{G}_L/D_{\mathfrak{p}}\}$. Since $\mathcal{G}_M\cap D_{\mathfrak{p}} = D_{\mathfrak{P}}$, we may regard, inside $\mathcal{O}[\mathcal{G}_L]\otimes_{\mathcal{O}[\mathcal{G}_M]} V$,

$$\begin{aligned}\sigma^{-1}\,\mathrm{Ind}_{\sigma D_{\mathfrak{P}}\sigma^{-1}}^{\sigma D_{\mathfrak{p}}\sigma^{-1}} V_{\sigma(\mathfrak{P})}^- &= \sigma^{-1}\mathcal{O}[\sigma D_{\mathfrak{p}}\sigma^{-1}]\otimes_{\mathcal{O}[\sigma D_{\mathfrak{P}}\sigma^{-1}]} V_{\sigma(\mathfrak{P})}^-\\ = \mathcal{O}[D_{\mathfrak{p}}\sigma^{-1}]\otimes_{\mathcal{O}[\sigma D_{\mathfrak{P}}\sigma^{-1}]} V_{\sigma(\mathfrak{P})}^- &\subset \mathcal{O}[D_{\mathfrak{p}}\sigma^{-1}\mathcal{G}_M]\otimes_{\mathcal{O}[\mathcal{G}_M]} V \subset \mathrm{Ind}_F^E V,\end{aligned} \tag{2.11}$$

where $V = V(\varphi_F)$ and $V^-_{\sigma(\mathfrak{P})} = V^-_{\sigma(\mathfrak{P})}(\varphi_F)$. We put

$$V^-_{\mathfrak{p}}(\mathrm{Ind}_F^E \varphi_F) = \sum_{\sigma} \sigma^{-1} \mathrm{Ind}_{\sigma D_{\mathfrak{p}}\sigma^{-1}}^{\sigma D_{\mathfrak{P}}\sigma^{-1}} V^-_{\sigma(\mathfrak{P})}, \tag{2.12}$$

which is stable under $D_{\mathfrak{p}}$. Thus $\{V^-_{\mathfrak{p}}(\mathrm{Ind}_F^E \varphi_F)\}$ gives a Selmer datum for $\mathrm{Ind}_F^E \varphi_F$. We write $\mathrm{Sel}(\mathrm{Ind}_F^E \varphi_F)$ for the Selmer group with respect to this datum.

Note that for $h \in \mathcal{G}_M$,

$$\begin{aligned}(h\sigma)^{-1} \mathrm{Ind}_{h\sigma D_{\mathfrak{p}}(h\sigma)^{-1}}^{h\sigma D_{\mathfrak{P}}(h\sigma)^{-1}} V^-_{h\sigma(\mathfrak{P})} &= (h\sigma)^{-1}\mathcal{O}[h\sigma D_{\mathfrak{p}}(h\sigma)^{-1}] \otimes_{\mathcal{O}[h\sigma D_{\mathfrak{P}}(h\sigma)^{-1}]} h(V^-_{\sigma(\mathfrak{P})}) \\ &= \sigma^{-1}\mathcal{O}[\sigma D_{\mathfrak{p}}\sigma^{-1}] \otimes_{\mathcal{O}[\sigma D_{\mathfrak{P}}\sigma^{-1}]} V^-_{\sigma(\mathfrak{P})} \\ &= \sigma^{-1} \mathrm{Ind}_{\sigma D_{\mathfrak{p}}\sigma^{-1}}^{\sigma D_{\mathfrak{P}}\sigma^{-1}} V^-_{\sigma(\mathfrak{P})} \subset \mathcal{O}[D_{\mathfrak{p}}\sigma^{-1}\mathcal{G}_M] \otimes_{\mathcal{O}[\mathcal{G}_M]} V.\end{aligned} \tag{2.13}$$

Then writing $\Delta_{\mathfrak{p}}$ for the image of $D_{\mathfrak{p}}$ in Δ, we conclude from (2.13)

$$\begin{aligned}V^-_{\mathfrak{p}}(\mathrm{Ind}_F^E \varphi_F) &= \bigoplus_{\sigma \in \mathcal{G}_M \backslash \mathcal{G}_L / D_{\mathfrak{p}}} \sigma^{-1} \mathrm{Ind}_{\sigma D_{\mathfrak{p}}\sigma^{-1}}^{\sigma D_{\mathfrak{P}}\sigma^{-1}} V^-_{\sigma(\mathfrak{P})} \\ &\cong \bigoplus_{\sigma \in \Delta/\Delta_{\mathfrak{p}}} \mathcal{O}[\Delta_{\mathfrak{p}}\overline{\sigma}^{-1}] \otimes_{\mathcal{O}} V^-_{\mathfrak{p}}(\varphi_E) = \mathcal{O}[\Delta] \otimes_{\mathcal{O}} V^-_{\mathfrak{p}}(\varphi_E),\end{aligned} \tag{2.14}$$

which shows $\mathrm{rank}_{\mathcal{O}} V^-_{\mathfrak{p}}(\mathrm{Ind}_F^E \varphi_F) = g(\mathfrak{P}/\mathfrak{p})e(\mathfrak{P}/\mathfrak{p})f(\mathfrak{P}/\mathfrak{p})d = [F:E]d$ for $d = \dim(V^-_{\mathfrak{p}}(\varphi_F))$ and by Shapiro's lemma,

$$\bigoplus_{\mathfrak{P}|\mathfrak{p}} H^1(I_{\mathfrak{P}}, V^-_{\mathfrak{P}}(\varphi_F)^*) \cong H^1(I_{\mathfrak{p}}, V^-_{\mathfrak{p}}(\mathrm{Ind}_F^E \varphi_F)^*).$$

Thus fixing an isomorphism: $V(\psi)^{m(\psi)} \cong \mathcal{O}[\Delta][\psi]$ for a model $V(\psi)$ of ψ, we define a Selmer datum for $\varphi \otimes \psi$:

$$V^-_{\mathfrak{p}}(\varphi_E \otimes \psi) = V(\psi) \otimes_{\mathcal{O}} V^-_{\mathfrak{p}}(\varphi_E) \subset \mathcal{O}[\Delta] \otimes_{\mathcal{O}} V(\varphi_E). \tag{2.15}$$

Then we get an isomorphism induced by ι

$$\iota : H^1(I_{\mathfrak{p}}, V^-(\mathrm{Ind}_F^E \varphi_F)^*) \cong \bigoplus_{\psi} H^1(I_{\mathfrak{p}}, V^-_{\mathfrak{p}}(\varphi_E \otimes \psi)^*)^{m(\psi)}. \tag{2.16}$$

We now look at a prime $\mathfrak{q} \nmid p$ of L ramifying in M/L. We have the following exact sequence:

$$\begin{aligned}0 \to H^1(I_{\mathfrak{q}}/I_{\mathfrak{Q}}, H^0(I_{\mathfrak{Q}}, V(\varphi_E \otimes \psi)^*) &\xrightarrow{cor} H^1(I_{\mathfrak{q}}, V(\varphi_E \otimes \psi)^*) \\ \xrightarrow{res} H^0(I_{\mathfrak{q}}, H^1(I_{\mathfrak{Q}}, V(\varphi_E \otimes \psi)^*)) &\xrightarrow{trans} H^2(I_{\mathfrak{q}}/I_{\mathfrak{Q}}, H^0(I_{\mathfrak{Q}}, V(\varphi_E \otimes \psi)^*).\end{aligned} \tag{2.17}$$

Since $I_{\mathfrak{q}}/I_{\mathfrak{Q}} \hookrightarrow \Delta$, the order of $I_{\mathfrak{q}}/I_{\mathfrak{Q}}$ is prime to p. Hence

$$H^j(I_{\mathfrak{q}}/I_{\mathfrak{Q}}, H^0(I_{\mathfrak{Q}}, V(\varphi_E \otimes \psi)^*)) = 0$$

for $j > 0$. This shows that any 1-cocycle u of $I_{\mathfrak{q}}$ trivial on $I_{\mathfrak{Q}}$ is already trivial on $I_{\mathfrak{q}}$, i.e., $u|_{I_{\mathfrak{Q}}} = 0 \Leftrightarrow u|_{I_{\mathfrak{q}}} = 0$. In particular, u is unramified at $\mathfrak{q}$.

By the above argument, we can define for each irreducible representation ψ of Δ,

$$\text{(2.18)} \quad \mathrm{Sel}(\varphi_E \otimes \psi)_{/L} = \mathrm{Ker}(H^1(\mathcal{G}_L, V(\varphi_E \otimes \psi)^*) \xrightarrow{r} \prod_{\mathfrak{q}\nmid p} H^1(I_{\mathfrak{q}}, V(\varphi_E \otimes \psi)^*) \times \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, V(\varphi_E \otimes \psi)^*/V_{\mathfrak{p}}^-(\varphi_E \otimes \psi)^*)).$$

Similarly we define

$$\text{(2.19)} \quad \mathrm{Sel}(\mathrm{Ind}_F^E \varphi_F)_{/L} = \mathrm{Ker}(H^1(\mathcal{G}_L, V(\mathrm{Ind}_F^E \varphi_F)^*) \xrightarrow{r} \prod_{\mathfrak{q}\nmid p} H^1(I_{\mathfrak{q}}, V(\mathrm{Ind}_F^E \varphi_F)^*) \times \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{q}}, V(\mathrm{Ind}_F^E \varphi_F)^*/V_{\mathfrak{p}}^-(\mathrm{Ind}_F^E \varphi_F)^*)).$$

Then the above argument shows the following isomorphism of Δ-modules:

(1) Applying Shapiro's lemma to local and global Galois groups,

$$\mathrm{Sel}(\mathrm{Ind}_F^E \varphi_F)_{/L} \cong \mathrm{Sel}(\varphi_F)_{/M} \cong \bigoplus_{\psi} \mathrm{Sel}(\varphi_F)_{/M}[\psi];$$

(2) $$\mathrm{Sel}(\mathrm{Ind}_F^E \varphi_F)_{/L} \cong \textstyle\bigoplus_{\psi} (\mathrm{Sel}(\varphi_E \otimes \psi)_{/E})^{m(\psi)}.$$

From this, we obtain

Proposition 2.4: *Let L/E and F/E be Galois extensions linearly disjoint over E. Suppose that $p \nmid [F:E]$. Put $M = LF$. Suppose near-ordinarity of type $\mathcal{F} = \{V(\overline{\delta}_{\mathfrak{p}})\}_{\mathfrak{p}|p}$ for $\overline{\rho}_E$: $\mathrm{Gal}(F^{(p,\infty)}/E) \to \mathrm{GL}_n(\mathbb{F})$. Let $\mathbb{J}$ be an integral domain in $\mathrm{CNL}_{\mathcal{O}}$. Then for each deformation φ_E: $\mathrm{Gal}(F^{p,\infty)}/E) \to \mathrm{GL}_n(\mathbb{J})$ nearly ordinary of type $\mathcal{F}$, we have an isomorphism of $\mathcal{O}[\Delta][\mathrm{Gal}(L/E)]$-modules:*

$$\mathrm{Sel}(\varphi_F)_{/M} \cong \mathrm{Sel}(\mathrm{Ind}_F^E \varphi_F)_{/L} \cong \bigoplus_{\psi} (\mathrm{Sel}(\varphi_E \otimes \psi)_{/L})^{m(\psi)},$$

where ψ runs over all irreducible representations of Δ and $m(\psi)$ is the multiplicity of ψ in $\mathcal{O}[\Delta]$. This isomorphism induces

$$\mathrm{Sel}(\varphi_F)_{/M}[\psi] \cong (\mathrm{Sel}(\varphi_E \otimes \psi)_{/L})^{m(\psi)},$$

where $\mathrm{Sel}(\varphi_F)_{/M}[\psi]$ is the ψ-isotypic component of $\mathrm{Sel}(\varphi_F)_{/M}$.

2.7. CONJECTURAL HOLOMORPHY OF THE p-ADIC L-FUNCTION OF $\mathrm{Ad}_S(\varphi) \otimes \psi$. The argument in Example 2.7 just tells us that the Selmer group $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi) \otimes \psi)_{/E}$ should be torsion under the assumptions in Example 2.7. On the p-adic L-function side, in principle, for a given absolutely irreducible Artin representation ψ: $\mathfrak{G} \to \mathrm{GL}_m(\mathbb{J})$, the p-adic L-function should be holomorphic at $s = 0$ if $\mathrm{Ad}_S(\varphi) \otimes \psi$ and $\mathrm{Ad}_S(\varphi) \otimes \psi\mathcal{N}$ for the p-adic cyclotomic character $\mathcal{N}$ (sending the geometric Frobenius $Frob_{\mathfrak{l}}$ to $N(\mathfrak{l})$) do not contains the trivial representation ([H96b] Section 4.4).

LEMMA 2.5: *Suppose $p \nmid [F : E]$. Then $\mathrm{Ad}_S(\overline{\rho}) \otimes \overline{\psi}$ does not contains the trivial character for an absolutely irreducible representation $\overline{\psi}$: $\Delta = \mathrm{Gal}(F/E) \to \mathrm{GL}_m(\mathbb{F})$ if (Z_F) is satisfied by $\overline{\rho}$. If $\mathrm{Ad}_S(\overline{\rho}) \otimes \overline{\psi}$ contains the Teichmüller character and $\overline{\rho}_F$ is absolutely irreducible, $\overline{\rho}_F$ is an induced representation from $F(\mu_p)$.*

Proof: The first assertion follows from the proof of Lemma 2.2. If $V(\mathrm{Ad}(\overline{\rho}_F)) \subset \mathrm{Hom}_{\mathbb{F}}(\overline{\rho}_F, \overline{\rho}_F)$ contains the Teichmüller character ω, then $\overline{\rho}_F \cong \overline{\rho}_F \otimes \omega$. This implies $\overline{\rho}_F \cong \mathrm{Ind}_{F(\mu_p)}^F \overline{\varepsilon}$ for a representation ε of $\mathfrak{H}_{F(\mu_p)}$, by [DHI] Lemma 3.2. ∎

By the lemma, under ($\mathrm{AI}_{F(\mu_p)}$), the cyclotomic p-adic L-function $L_p(s, \mathrm{Ad}_S(\varphi) \otimes \psi)$ (if it exists) should be holomorphic at $s = 0$.

3. Base-change for deformation rings

Let E be a number field or a p-adic field field with (p-adic) integer ring O_E. Let F/E be a finite Galois extension with Galois group Δ. Let $M = \overline{E}$ if E is a p-adic field and $M = F^{(p,\infty)}$ if E is a number field. We write $\mathfrak{G} = \mathrm{Gal}(M/E)$ and $\mathfrak{H} = \mathrm{Gal}(M/F)$. Then we fix a n-dimensional continuous Galois representation $\overline{\rho}$: $\mathfrak{G} \to G(\mathbb{F})$ for a finite field $\mathbb{F}$ of characteristic p. Let $n = \dim \overline{\rho}$ and suppose that $p \nmid 2n$. We suppose that $\overline{\rho}$ is nearly ordinary of type $\mathcal{F} = \{P_{\mathfrak{p}}\}_{\mathfrak{p}}$ if E is a number field. We study the relation between deformation functors of $\overline{\rho}$ on $\mathfrak{G}$ and $\mathfrak{H}$.

3.1. DEFORMATION FUNCTORS. Let L/E be a subextension of M/E with $\mathfrak{H}_L = \mathrm{Gal}(M/L)$. Fix two characters $\phi_{\det}$: $\mathfrak{G} \to \mathcal{O}^\times$ and ϕ_ν: $\mathfrak{G} \to \mathcal{O}^\times$ such that $\phi_{\det} \equiv \det \circ \overline{\rho} \bmod \mathfrak{m}_{\mathcal{O}}$, $\phi_\nu \equiv \nu \circ \overline{\rho} \bmod \mathfrak{m}_{\mathcal{O}}$ and $\phi_{\det}^2 = \phi_\nu^n$. Since $p \nmid 2n$, the information of ϕ_ν determines $\phi_{\det}$ and vice versa. When $G = \mathrm{GL}(n)$, we disregard ϕ_ν. We study the following functors defined on $\mathrm{CNL}_{\mathcal{O}}$:

$$(3.1)\quad \Phi_{G,L}^{full}(A) = \{\rho: \mathfrak{H}_L \to G(A) \mid \rho \equiv \overline{\rho} \bmod \mathfrak{m}_A\} / \approx,$$

(3.2) $\Phi^{n.ord}_{G,L}(A) = \{\rho\colon \mathfrak{H}_L \to G(A) \in \Phi^{full}(A) |\ \rho \text{ is nearly ordinary of type } \mathcal{F}\}$,

(3.3) $\Phi^{\phi}_{G,L}(A) = \{\rho\colon \mathfrak{H}_L \to G(A) \in \Phi^{n.ord}(A) |\ \det\circ\rho = \phi_{\det} \text{ and } \nu\circ\rho = \phi_\nu\}$.

Hereafter we fix G and write $\Phi^?_L$ for the functor defined above. Of course, $\Phi^{n.ord}_L$ is defined only for global L.

Remark 3.1: We introduce one more functor to cover Galois representation associated to unitary groups. Let c be an element of order 2 in $\mathrm{Aut}(E)$. Extends c to M. Then c acts on $\mathfrak{G}$ by $g \mapsto cgc^{-1}$. We consider $\rho^c(g) = \rho(cgc^{-1})$ for $\rho \in \Phi^{full}_L(A)$ as long as c leaves L stable. We suppose that $\overline{\rho}^c \cong \tilde{\overline{\rho}}$. Then we define a functor, supposing the properties: ? are stable under $\rho \mapsto \widetilde{\rho}^c$,

$$\Phi^?_{U,L}(A) = \left\{\rho \in \Phi^?_{\mathrm{GL}(n),L}(A) \middle|\ \rho^c \cong \widetilde{\rho}\right\},$$

where ? is a combination of ϕ, $full$ and $n.ord$. This functor is representable by $R^\phi_{U,L} \in \mathrm{CNL}_\mathcal{O}$ under (Z_L) if ? $= full$ and ? $= (full, \phi)$ and under (Z_L) and (Reg_L) if ? $= n.ord$ and ? $= \phi$. By definition, the functorial map: $\rho \mapsto \widetilde{\rho}^c$ induces an involution on $\Phi^?_{\mathrm{GL}(n),L}$ and hence on the universal deformation ring $R^?_{\mathrm{GL}(n),L}$ representing the functor $\Phi^?_{\mathrm{GL}(n),L}$ and on the corresponding Selmer group. We write the fixed part of the involution of $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))_{/L}$ as $\mathrm{Sel}^*_{ac}(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))_{/L}$ and call it the anti-cyclotomic part of $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))_{/L}$. Then if $p \nmid 2n$, we have

$$\Omega_{R^\phi_{U,L}/R^I_{\mathrm{GL}(n),L}} \otimes_{R^\phi_{U,L}} \mathbb{J} \cong \mathrm{Sel}^*_{ac}(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))_{/L}. \tag{3.4}$$

3.2. Base-change. Let L/L' be an intermediate Galois extension of M/E with Galois group Γ. Suppose that $\Phi^?_{G,L}$ and $\Phi^?_{G,L'}$ are both representable. Let $\alpha^? = \alpha^?_{L/L'} : R^?_{G,L} \to R^?_{G,L'}$ be the base change morphism defined by $\alpha^?\varrho^?_L \approx \varrho^?_{L'}|_{\mathfrak{H}_L}$ for $\mathfrak{H}_L = \mathrm{Gal}(M/L)$. We like to study the morphism $\alpha^?$, in particular, its kernel and cokernel. For that, we recall the theory of I. Schur describing when one can extend a representation of a normal subgroup to the ambient group (see [H96a] Appendix).

We recall the condition (Sm) in 2.3 that G is smooth over $\mathcal{O}$, and by that, we have

(S) *The reduction map: $G(\mathcal{O}) \to G(\mathbb{F})$ is surjective.*

For each $\sigma \in \mathfrak{H}_{L'}$, we choose $\ell(\sigma) \in G(\mathcal{O})$ such that $\ell(\sigma) \equiv \overline{\rho}(\sigma) \bmod \mathfrak{m}_\mathcal{O}$. Then we define $\rho^\sigma(g) = \ell(\sigma)^{-1}\rho(\sigma g\sigma^{-1})\ell(\sigma)$. The strict equivalence class $[\rho^\sigma] \in \Phi^?_L(A)$ is independent of the choice of $\ell(\sigma)$. Thus $\mathfrak{H}_{L'}$ acts on $\Phi^?_{L'}(A)$. If $\sigma \in \mathfrak{H}_L$, then

$$\rho^\sigma(g) = \ell(\sigma)^{-1}\rho(\sigma g\sigma^{-1})\ell(\sigma) = \ell(\sigma)^{-1}\rho(\sigma)\rho(g)\rho(\sigma^{-1})\ell(\sigma),$$

and $\ell(\sigma)^{-1}\rho(\sigma) \in \widehat{G}(A)$. Therefore, the action factors through $\Gamma = \mathrm{Gal}(L/L')$.

Suppose (Z_L) and $\rho \in \Phi_L^{?,\Gamma}(A) = \{\rho \in \Phi_L^?(A) | \rho^\sigma \approx \rho \text{ for all } \sigma \in \Gamma\}$. Then we can find a map $c: \mathfrak{H}_{L'} \to G(A)$ such that $c \equiv \overline{\rho} \bmod \mathfrak{m}_A$ and $\rho = c(\sigma)^{-1}\rho^\sigma c(\sigma)$. We may further assume that $c(1) = 1$ and $c(h\sigma) = \rho(h)c(\sigma)$ if $h \in \mathfrak{H}_L$ (see [H96a] Appendix). Then we define $b(\sigma,\tau) = b_\rho(\sigma,\tau)$: $\mathfrak{H}_{L'} \times \mathfrak{H}_{L'} \to G(A)$ by

$$c(\sigma)c(\tau) = b(\sigma,\tau)c(\sigma\tau).$$

Since b has values in the centralizer $Z(\rho)$ of ρ, b has values in $\widehat{G}(A) \cap Z(A) = \widehat{\mathbf{G}_m}(A)$ by (Z_L) for the center $Z(A) = A^\times$ of G. As seen in [H96a] page 116, b factors through Γ. Thus b is a 2-cocycle of Γ having values in $\widehat{\mathbf{G}_m}(A)$. The cohomology class $[\rho] = [b_\rho] \in H^2(\Gamma, \widehat{\mathbf{G}_m}(A))$ is well defined independently of the choice of c (see [H96a] Appendix). If b_ρ is a coboundary, writing $b_\rho(\sigma,\tau) = \partial\zeta(\sigma,\tau) = \zeta(\sigma)\zeta(\tau)\zeta(\sigma\tau)^{-1}$ by a cochain ζ: $\mathfrak{H}_{L'} \to \widehat{\mathbf{G}_m}(A)$, $\pi : \zeta(\sigma)^{-1}c(\sigma)$ is a representation of $\mathfrak{H}_{L'}$ into $G(A)$ extending ρ. On the other hand, if there exists an extension π: $\mathfrak{H}_L \to G(A)$, then b_ρ is a coboundary of $\zeta(\sigma) = \pi(\sigma)^{-1}c(\sigma) \in \widehat{\mathbf{G}_m}(A)$. Thus

$$[\rho] = 0 \iff \rho \text{ extends to a representation } \pi\colon \mathfrak{H}_{L'} \to G(A).$$

If $G = \mathrm{GL}(n)$, $b_\rho^n = \det(b_\rho) = b_{\det(\rho)}$. Thus $[\det(\rho)] = n[\rho]$, and extensibility of ρ is equivalent to that of $\det(\rho)$ under $p \nmid n$. In particular, if $\det(\rho) = \phi_{\det}$, the extension $\phi_{\det}$ corresponds to a 1-cochain ζ such that $b_{\det(\rho)} = \partial\zeta$. Then $b_\rho = \partial\zeta^{1/n}$. The extension $\pi = \zeta^{-1/n}c$ then satisfies $\det(\pi) = \phi_{\det}$. Moreover such extension is unique, because all other extension is given by $\pi \otimes \chi$ for a character χ: $\Gamma \to A^\times$ (see [H96a] Appendix). As seen in [H96a] Section A.2.2, under (Reg_L), the extension of $\rho \in \Phi_L^{n.ord}(A)$ is again nearly ordinary of type $\mathcal{F}$. Thus under the assumption that $p \nmid n$, we can extend $\rho \in \Phi^\phi_{\mathrm{GL}(n),L}(A)$ uniquely to an element of $\Phi^\phi_{\mathrm{GL}(n),L'}(A)$ giving

$$\Phi^\phi_{\mathrm{GL}(n),L'}(A) \cong \Phi^{\phi,\Gamma}_{\mathrm{GL}(n),L}(A).$$

If $G \neq \mathrm{GL}(n)$, writing $(\ ,\ )_A$ for the pairing defining the similitude group $G(A)$, we see

$$\begin{aligned} b_\rho(\sigma,\tau)^2(x,y)_A &= (b(\sigma,\tau)x, b(\sigma,\tau)y)_A \\ &= (c(\sigma)c(\tau)c(\sigma\tau)^{-1}x, c(\sigma)c(\tau)c(\sigma\tau)^{-1}y)_A \\ &= \nu(c(\sigma)c(\tau)c(\sigma\tau)^{-1})(x,y)_A. \end{aligned}$$

This shows, in $H^2(\Gamma, \widehat{\mathbf{G}_m}(A))$,

$$2[\rho] = [\nu\rho],$$

where $\nu\rho = \nu \circ \rho$: $\mathfrak{H}_{L'} \to \mathbf{G}_m(A)$ is the similitude character of ρ. Suppose that $[\nu\rho] = 0$. Then, we can find ζ: $\Gamma \to \mathbf{G}_m(A)$ such that $b_\rho(\sigma,\tau) = \zeta(\sigma)\zeta(\tau)\zeta(\sigma\tau)^{-1}$ and $b_{\nu\rho}(\sigma,\tau) = \zeta(\sigma)^2\zeta(\tau)^2\zeta(\sigma\tau)^{-2}$. Then $\pi(g) = c(g)\zeta(g)^{-1}$ is an extension of ρ with $\nu\pi = \nu c\zeta^{-2}$. Similarly, we get $b_{\det(\rho)} = \zeta(\sigma)^n\zeta(\tau)^n\zeta(\sigma\tau)^{-n}$. Thus if $p \nmid 2n$, then we have

$$\Phi^{\phi}_{G,L'}(A) \cong \Phi^{\phi,\Gamma}_{G,L}(A).$$

If $[L : L']$ is prime to p, the obstruction cohomology class $[\rho]$ always vanishes, because $H^2(\Gamma, \widehat{\mathbf{G}_m}(A)) = 0$. Thus we have

$$\Phi^{?}_{G,L'} \cong \Phi^{?,\Gamma}_{G,L}.$$

Summing up, we get

PROPOSITION 3.1: *Suppose $p \nmid 2n$, (Z_L) and* (Sm). *When E is a number field, we suppose* (Reg_L) *and $(\mathrm{Z}_{p,L})$ in the nearly ordinary case. Then, if $p \nmid [L : L']$, we have*

$$\Phi^{?}_{G,L'} \cong \Phi^{?,\Gamma}_{G,L}(A)$$

for $? = full, (full,\phi), n.ord, \phi$. Suppose that $\nu(\rho)$ or $\det(\rho)$ *can be extended to a character ϕ: $\mathfrak{H}_{L'} \to \mathbf{G}_m(A)$. Then (even if $p|[L : L']$), each $\rho \in \Phi^{\phi,\Gamma}_{G,L'}(A)$ can be extended to a representation π: $\mathfrak{H}_{L'} \to G(A)$ such that $\pi \equiv \overline{\rho}_{L'} \bmod \mathfrak{m}_A$ and $\nu\pi = \phi$. Moreover under* (Reg_L), *we have*

$$\Phi^{\phi}_{G,L'} \cong \Phi^{\phi,\Gamma}_{G,L} \quad \text{and} \quad \Phi^{\phi,full}_{G,L'} \cong \Phi^{\phi,full,\Gamma}_{G,L},$$

where

$$\Phi^{\phi,full}_{G,L}(A) = \{\rho \in \Phi^{full}_{G,L}(A) | \ (\det(\rho), \nu(\rho)) = \phi\}.$$

Let $\mathcal{F}_L$: $\mathrm{CNL}_{\mathcal{O}} \to SETS$ be the full deformation functor on $\mathfrak{H}_L$ of the character $\det(\overline{\rho}_L)$. For each $\rho \in \Phi^{n.ord}_{G,L}(A)$, $\det(\rho)^{-1}\phi_{\det}$ has values in $\widehat{\mathbf{G}_m}(A)$. Thus if $p \nmid n$, $(\det(\rho)^{-1}\phi_{\det})^{1/n}$ is uniquely determined. If further $p \nmid 2n$, $\rho \otimes (\det(\rho)^{-1}\phi_{\det})^{1/n}$ is an element in $\Phi^{\phi}_{G,L}(A)$, and we may associate $(\rho \otimes (\det(\rho)^{-1}\phi_{\det})^{1/n}, \det(\rho)) \in \Phi^{\phi}_{L}(A) \times \mathcal{F}_L(A)$ to $\rho \in \Phi^{n.ord}(A)$. Since we can recover ρ out of the pair (ρ_ϕ, ξ) by $\rho = \rho_\phi \otimes (\phi_{\det}^{-1}\xi)^{1/n}$, we have

$$\Phi^{n.ord}_{G,L} \cong \Phi^{\phi}_{L} \times \mathcal{F}_L \quad \text{and} \quad \Phi^{full}_{G,L} \cong \Phi^{full,\phi}_{L} \times \mathcal{F}_L.$$

This shows

COROLLARY 3.2: *Suppose $p \nmid 2n$, (Z_L) and* (Sm). *When E is a number field, we suppose* (Reg_L) *and $(\mathrm{Z}_{p,L})$ in the nearly ordinary case. Then*

(1) *If $p \nmid [L : L']$, the base change morphism $\alpha^?$ induces*

$$R^?_{L,G}/\sum_{\gamma\in\Gamma} R^?_{L,G}(\gamma-1)R^?_{L,G} \cong R^?_{L',G}$$

for $? = full, (full, \phi), n.ord, \phi$.

(2) *Even if $p|[L : L']$, the base change morphism $\alpha^?$ induces*

$$R^{\phi}_{L,G}/\sum_{\gamma\in\Gamma} R^{\phi}_{L,G}(\gamma-1)R^{\phi}_{L,G} \cong R^{\phi}_{L',G},$$

$$R^{\phi,full}_{L,G}/\sum_{\gamma\in\Gamma} R^{\phi,full}_{L,G}(\gamma-1)R^{\phi,full}_{L,G} \cong R^{\phi,full}_{L',G} \quad \text{and}$$

$$R^?_{L,G}/\sum_{\gamma\in\Gamma} R^?_{L,G}(\gamma-1)R^?_{L,G} \cong \mathrm{Im}(\alpha^?_{L'/L}),$$

where $? = full$ and $n.ord$. Moreover we have a canonical isomorphism:

$$R^?_{L,G} \cong \mathrm{Im}(\alpha^?_{L/L'})\widehat{\otimes}_{\Lambda_L}\Lambda'_L,$$

where $? = full$ and $n.ord$, and Λ_L is the universal deformation ring representing $\mathcal{F}_L$. The ring Λ_L is isomorphic to the Iwasawa algebra $\mathcal{O}[[\mathfrak{H}^{ab}_{L,p}]]$ for the maximal p-profinite abelian quotient $\mathfrak{H}^{ab}_{L,p}$ of $\mathfrak{H}_L$.

Proof: The assertion (1) and the first assertion of (2) follow directly from Proposition 3.1. By the argument just prior to the corollary, we see $R^{n.ord}_{L,G} \cong R^{\phi}_{L,G}\widehat{\otimes}_{\mathcal{O}}\Lambda_L$. Under this decomposition, $\alpha^{n.ord}_{L/L'} = \alpha^{\phi}_{L/L'} \otimes \beta_{L/L'}$, where $\beta_{L/L'}$: $\Lambda_L \to \Lambda'_L$ is the base change map associated to $\mathcal{F}_{L'}(A) \ni \xi \mapsto \xi|_{\mathfrak{H}_L} \in \mathcal{F}_L(A)$. The base change map $\beta_{L/L'}$ is induced by the inclusion: $\mathfrak{H}_L \hookrightarrow \mathfrak{H}_{L'}$. From the first assertion of (2), we get the second and

$$R^{n.ord}_{L',G} \cong R^{\phi}_{L,G}\widehat{\otimes}_{\mathcal{O}}\Lambda_L \cong \mathrm{Im}(\alpha^{\phi}_{L/L'})\widehat{\otimes}_{\mathcal{O}}(\mathrm{Im}(\beta_{L/L'})\widehat{\otimes}_{\Lambda_L}\Lambda_{L'}) \cong \mathrm{Im}(\alpha^{n.ord}_{L/L'})\widehat{\otimes}_{\Lambda_L}\Lambda_{L'},$$

which finishes the proof. ∎

3.3. Control of Kähler differentials. We fix a $\mathbb{Z}_p$-extension $F_{\infty/F}$ with $\Gamma = \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$. Later we specialize our argument to the case where $F_\infty = FE_\infty$ for a $\mathbb{Z}_p$-extension E_∞/E, but for the moment, F_∞/F is an arbitrary $\mathbb{Z}_p$-extension. Let $\mathfrak{H}_\infty = \mathrm{Gal}(M/F_\infty)$; hence, $\mathfrak{G}/\mathfrak{H}_\infty = \Gamma$. Let $\bar\rho$: $\mathfrak{G} \to G(\mathbb{F})$ be a representation. For $\bar\rho$, we suppose (Z_F), (Reg_F) and $(\mathrm{Z}_{p,F})$ in the nearly ordinary global case. In the nearly ordinary case, we also suppose:

(EP) $$p \nmid 2n\prod_{j,\mathfrak{p}} \dim\bar\delta_{j,\mathfrak{p}}.$$

Under these conditions, the functors $\Phi^?_F$ studied in the prior subsection are representable. Then by Lemma 2.2, the functor $\Phi^?_j = \Phi^?_{F_j}$ is again representable for the j-th layer F_j/F. Over F_∞, the functor $\Phi^?_\infty$ may not be representable but at least pro-representable. We write $(R^?_{F_j}, \varrho^?_{F_j})$ for the universal couple representing $\Phi^?_j$.

We apply Corollary 3.2 to $F_\infty/F_j/F$. We write γ_j for a generator of $\Gamma_j = \mathrm{Gal}(F_\infty/F_j)$. Let $R^?_\infty = R^?_{F_\infty} = \varprojlim_j R^?_{F_j}$ for $? = full$, $n.ord$ or ϕ. Then $R^?_\infty$ prorepresents $\Phi^?_{F_\infty}$. We then put $R^?_j = \alpha_{F_\infty/F_j}(R^?_\infty) = \mathrm{Im}(\alpha_{F_\infty/F_j}) \subset R^?_{F_j}$. Then by Corollary 3.2, we have $R^\phi_j = R^\phi_{F_j}$ and

$$R^?_j \cong R^?_\infty / R^?_\infty(\gamma_j - 1)R^?_\infty. \tag{3.5}$$

Proposition 3.3: *Let the notation be as above. Suppose that $? = \phi$, full or n.ord. Let A be a closed $\mathcal{O}$-subalgebra of $R^?_\infty$ (in $\mathrm{CNL}_\mathcal{O}$) on which Γ acts trivially. Let B be an A-algebra in $\mathrm{CNL}_\mathcal{O}$ and π: $R^?_0 \to B$ be an A-algebra homomorphism. Then we have for $0 \le j \le k \le \infty$,*

$$\frac{\Omega_{R^?_k/A}\widehat{\otimes}_{R^?_k} B}{(\gamma_j - 1)\Omega_{R^?_k/A}\widehat{\otimes}_{R^?_k} B} \cong \Omega_{R^?_j/A}\widehat{\otimes}_{R^?_j} B.$$

Although this result is intuitive and is essentially deduced from Corollary 3.2 in [H96a] Corollary 1.1, we shall give a proof since this is fundamental in the sequel:

Proof: We write R for $R^?_k\widehat{\otimes}_A B$ and R' for $R^?_j\widehat{\otimes}_A B$. Then $R/R(\gamma_j - 1)R \cong R'$. Write α for the projection: $R \to R'$ and π' for $m\circ((\pi\circ\alpha^?_j)\otimes\mathrm{id}) : R' = R^?_k\widehat{\otimes}_A B \to B\widehat{\otimes}_A B \overset{m}{\to} B$ for multiplication m: $B\widehat{\otimes}_A B \to B$. Let $\lambda = \pi' \circ \alpha$. We have

$$\begin{aligned}\mathrm{Ker}(\lambda) \otimes_R B = \mathrm{Ker}(\lambda)/\mathrm{Ker}(\lambda)^2 = \Omega_{R/B} \otimes_R B = \Omega_{R^?_k\widehat{\otimes}_A B/A\otimes_A B} \otimes_R B \\ \cong (\Omega_{R^?_k/A}\widehat{\otimes}_A B) \otimes_R B \cong \Omega_{R^?_k/A}\widehat{\otimes}_{R^?_k} B.\end{aligned}$$

Similarly, we have $\mathrm{Ker}(\pi') \otimes_R B \cong \Omega_{R^?_j/A}\widehat{\otimes}_{R^?_j} B$. We have the following exact sequence:

$$0 \to R(\gamma_j - 1)R \to \mathrm{Ker}(\lambda) \overset{\alpha}{\to} \mathrm{Ker}(\pi') \to 0. \tag{3.7}$$

Tensoring B over R to (3.7) and writing $J = R(\gamma_j - 1)R$, we get another exact sequence:

$$(J/J^2) \otimes_R B = J \otimes_R B \overset{i}{\to} \Omega_{R^?_k/A}\widehat{\otimes}_{R^?_k} B \longrightarrow \Omega_{R^?_j/A}\widehat{\otimes}_{R^?_j} B \longrightarrow 0.$$

We look into the B-linear map $\gamma_j - 1\colon \mathrm{Ker}(\lambda) \to R$. Write B' for the image of B in R. Then $B' \subset \mathrm{Ker}(\gamma_j - 1)$ and $R = \mathrm{Ker}(\lambda) + B'$. Thus $(\gamma_j - 1)R = (\gamma_j - 1)\,\mathrm{Ker}(\lambda)$. Since γ_j is a B'-algebra automorphism of R and J/J^2 is a B'-module, we have

$$r(\gamma_j - 1)r' \equiv (\gamma_j - 1)rr' \bmod J^2 \quad (r, r' \in J).$$

This shows that $\gamma_j - 1\colon \mathrm{Ker}(\lambda) \to R$ induces a surjective morphism of B'-modules: $\mathrm{Ker}(\lambda)/\mathrm{Ker}(\lambda)^2 \to J/J^2$; thus, $\mathrm{Im}(i) = (\gamma_j - 1)(\Omega_{R_k^?/A} \widehat{\otimes}_{R_k^?} B)$, which shows the result. ∎

We can slightly generalize the above result as follows:

Corollary 3.4: *Let the notation be as above. Suppose that $? = \phi$, full or n.ord. Let A_∞ be an $\mathcal{O}$-algebra with a continuous Γ-action which is a pro-object of $\mathrm{CNL}_\mathcal{O}$. We suppose that R_∞ has a structure of A_∞-algebra and that the Γ-action on A_∞ and R_∞ is compatible. Thus $R_j^?$ is an A_j-algebra for $A_j = A_\infty/A_\infty(\gamma_j - 1)A_\infty$. Let B be an A_∞-algebra in $\mathrm{CNL}_\mathcal{O}$ and $\pi\colon R_0^? \to B$ be an A_∞-algebra homomorphism. Then we have for $0 \le j \le k \le \infty$,*

$$\frac{\Omega_{R_k^?/A_k} \widehat{\otimes}_{R_k^?} B}{(\gamma_j - 1)\Omega_{R_k^?/A_k} \widehat{\otimes}_{R_k^?} B} \cong \Omega_{R_j^?/A_j} \widehat{\otimes}_{R_j^?} B.$$

Proof: By the assumption and the proof of Proposition 3.3, we have

$$\frac{\Omega_{R_k^?/\mathcal{O}} \widehat{\otimes}_{R_k^?} B}{(\gamma_j - 1)\Omega_{R_k^?/\mathcal{O}} \widehat{\otimes}_{R_k^?} B} \cong \Omega_{R_j^?/\mathcal{O}} \widehat{\otimes}_{R_j^?} B$$

and

$$\frac{\Omega_{A_k/\mathcal{O}} \widehat{\otimes}_{A_k} B}{(\gamma_j - 1)\Omega_{A_k/\mathcal{O}} \widehat{\otimes}_{A_k} B} \cong \Omega_{A_j/\mathcal{O}} \widehat{\otimes}_{A_j} B.$$

This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\dfrac{\Omega_{A_\infty/\mathcal{O}} \otimes B}{(\gamma_j - 1)\Omega_{A_\infty/\mathcal{O}} \otimes B} & \longrightarrow & \dfrac{\Omega_{R_\infty^?/\mathcal{O}} \otimes B}{(\gamma_j - 1)\Omega_{R_\infty^?/\mathcal{O}} \otimes B} & \longrightarrow & \dfrac{\Omega_{R_\infty^?/A_\infty} \otimes B}{(\gamma_j - 1)\Omega_{R_\infty^?/A_\infty} \otimes B} & \longrightarrow & 0 \\
\wr\Vert\downarrow & & \wr\Vert\downarrow & & \downarrow & & \\
\Omega_{A_j/\mathcal{O}} \otimes B & \longrightarrow & \Omega_{R_j^?/\mathcal{O}} \otimes B & \longrightarrow & \Omega_{R_j^?/A_j} \otimes B & \longrightarrow & 0.
\end{array}$$

We then conclude

$$\frac{\Omega_{R_\infty^?/A_\infty} \otimes B}{(\gamma_j - 1)\Omega_{R_\infty^?/A_\infty} \otimes B} \cong \Omega_{R_j^?/A_j} \otimes B,$$

which finishes the proof. ∎

4. Control theory for Selmer groups

We return to the situation in Section 2. Thus F/E is an extension of a number field E, $\mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E)$ and $\mathfrak{H} = \mathrm{Gal}(F^{(p,\infty)}/F)$. We fix a Galois representation $\varphi: \mathfrak{G} \to G(\mathbb{J}) \subset \mathrm{GL}_n(\mathbb{J})$ nearly ordinary of type $\mathcal{F} = \{(\mathrm{fil}_\mathfrak{p})\}_{\mathfrak{p}|p}$. Let $\overline{\rho}: \mathfrak{G} \to G(\mathbb{F})$ be the residual representation of φ. We assume that

(EP) *p is prime to $2n[F:E]$ and* $\dim \overline{\delta}_{j,\mathfrak{p}}$ *for all* $\mathfrak{p}|p$ *and* j

in addition to (Z_F), $(\mathrm{Z}_{p,F})$ and (Reg_F). Let F_∞/F be a $\mathbb{Z}_p$-extension such that

(TR) $F_{\infty,\mathfrak{p}} = \bigcup_j F_{j,\mathfrak{p}}$ is a totally ramified $\mathbb{Z}_p$-extension of $F_\mathfrak{p}$ for all $\mathfrak{p}|p$.

4.1. Global control theory. For $H = I$ or D, we put $R^H_\infty = R^H_{G,F_\infty}$ in (2.2). Then we put R^H_j to be the image under the base change map in $R^H_{F_j}$ of R^H_∞. Then by Corollary 3.2, we have locally and globally

$$R^H_j \cong R^H_\infty/R^H_\infty(\gamma_j - 1)R^H_\infty \quad \text{and} \quad R^\phi_j \cong R^\phi_\infty/R^\phi_\infty(\gamma_j - 1)R^\phi_\infty. \tag{4.1}$$

Thus we have from Corollary 3.4

$$\frac{\Omega_{R^\phi_\infty/R^H_\infty} \otimes \mathbb{J}}{(\gamma_j - 1)\Omega_{R^\phi_\infty/R^H_\infty} \otimes \mathbb{J}} \cong \Omega_{R^\phi_j/R^H_j} \otimes \mathbb{J}. \tag{4.2}$$

Applying this to $H = I$, we get

Proposition 4.1:

$$\frac{\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi) \otimes \kappa)_{/F}}{(\gamma_j - 1)\,\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi) \otimes \kappa)_{/F}} \cong \Omega_{R^\phi_{F_j}/R^I_j} \otimes \mathbb{J}.$$

We note here that we have the following exact sequences:

$$\Omega_{R^I_F/R^I_0} \otimes_{R^I_F} \mathbb{J} \to \Omega_{R^\phi_F/R^I_0} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F} \to 0, \tag{4.3}$$

$$\Omega_{R^D_0/R^I_0} \otimes_{R^D_0} \mathbb{J} \to \Omega_{R^\phi_F/R^I_0} \otimes \mathbb{J} \to \Omega_{R^\phi_F/R^D_0} \otimes \mathbb{J} \to 0. \tag{4.4}$$

4.2. Local contribution. We now make explicit the Δ-module $\Omega_{R^I_F/R^I_0} \otimes_{R^I_F} \mathbb{J}$ and $\Omega_{R^D_0/R^I_0} \otimes_{R^D_0} \mathbb{J}$. Since G is split over $\mathcal{O}$, the center $ZM_\mathfrak{p}$ of the standard Levi subgroup $M_\mathfrak{p}$ of the parabolic subgroup $P_\mathfrak{p} \subset G$ is isomorphic to $\mathbf{G}_m^{m_\mathfrak{p}}$. Then the respresentation $\overline{\delta}_\mathfrak{p}: D_\mathfrak{p} \to M_\mathfrak{p}$ can be regarded as a product of $m_\mathfrak{p}$-absolutely irreducible representations $\overline{\delta}_{i,\mathfrak{p}}$ for $i = 1, 2, \ldots, m_\mathfrak{p}$. We can thus split

$$R^I_j = \widehat{\bigotimes}_{\mathfrak{p} \in S_{F_j}} \widehat{\bigotimes}_{i=1}^{m_\mathfrak{p}} R^I_{j,\overline{\delta}_{i,\mathfrak{p}}}$$

into the tensor product of components $R^I_{j,\overline{\delta}_{i,\mathfrak{p}}}$ associated to $\overline{\delta}_{i,\mathfrak{p}}$. The component $R^I_{j,\overline{\delta}_{i,\mathfrak{p}}}$ is the image under the base change map of the universal deformation ring of the representation $\overline{\delta}_{i,\mathfrak{p}}$ of the ineratia group at $\mathfrak{p}$ over $F_{\infty,\mathfrak{p}} = \bigcup_j F_{j,\mathfrak{p}}$ in the universal deformation ring $R^I_{\overline{\delta}_{i,\mathfrak{p}},F_j}$ of the inertia group over $F_{j,\mathfrak{p}}$. By Corollary 3.2, we see

$$R^I_{\overline{\delta}_{i,\mathfrak{p}},F_j} \cong R^I_{j,\overline{\delta}_{i,\mathfrak{p}}} \widehat{\otimes}_{\Lambda_{I_{\infty,I_\mathfrak{p}}}} \Lambda_{I_{j,\mathfrak{p}}},$$

where $\Lambda_{I_{j,\mathfrak{p}}} = \mathcal{O}[[I^{ab}_{j,\mathfrak{p},p}]]$ for the maximal p-profinite abelian quotient $I^{ab}_{j,\mathfrak{p},p}$ of the inertia subgroup $I_{j,\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_{j,\mathfrak{p}})$ for $j = 0, 1, \ldots \infty$. Note that the natural algebra homomorphism of $R^I_{\overline{\delta}_{i,\mathfrak{p}},F}$ into R^ϕ_F factors through the universal deformation ring $R^D_{\overline{\delta}_{i,\mathfrak{p}},F}$ of $\overline{\delta}_{i,\mathfrak{p}}$ over $D_\mathfrak{p}$. Thus by applying the base change map α, we get that the image of $\Lambda_{I_\infty,\mathfrak{p}}$ in $R^D_{\overline{\delta}_{i,\mathfrak{p}},F}$ is, by local class field theory, isomorphic to $\mathcal{O}[[W]]$ for the universal norm group W inside $O^\times_{F_\mathfrak{p}}$. Then by the total ramification of $F_{\infty,\mathfrak{p}}/F_\mathfrak{p}$, $O^\times_{F_\mathfrak{p}}/W \cong \Gamma$. The natural map of $\Lambda_{I_{0,\mathfrak{p}}}$ into $R^D_{\overline{\delta}_{i,\mathfrak{p}},F}$ is given by the determinant character $\det \delta_{i,\mathfrak{p},\varrho}$ which factors through the image (isomorphic to $O^\times_{F_\mathfrak{p}}$) of $I_\mathfrak{p}$ in $D^{ab}_{\mathfrak{p},p}$. Thus the image of $\Omega_{R^I_{\xi,F}/R^I_{0,\overline{\xi}}} \otimes_{R^I_{\xi,F}} \mathbb{J}$ in (4.3) is equal to the image of $\Omega_{\mathcal{O}[[O^\times_\mathfrak{p}]]/\mathcal{O}[[W]]} \otimes_{\mathcal{O}[[O^\times_\mathfrak{p}]]} \mathbb{J} \cong \Gamma \otimes_{\mathbb{Z}_p} \mathbb{J} \cong \mathbb{J}$. In this way, we get from (4.3) the following exact sequence:

$$(4.5) \qquad \bigoplus_{\mathfrak{p}\in S_E} \mathbb{J}[S_\mathfrak{p}]^{m_\mathfrak{p}-1} \xrightarrow{\iota_I} \Omega_{R^\phi_F/R^I_0} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F} \to 0,$$

where $S_\mathfrak{p}$ is the set of primes of F over $\mathfrak{p} \in S_E$ in E and $\mathbb{J}[S_\mathfrak{p}]$ is the $\mathbb{J}$-free module generated by the elements of $S_\mathfrak{p}$, on which $\Delta = \mathrm{Gal}(F/E)$ acts by its action on the set $S_\mathfrak{p}$. Here we have the exponent $m_\mathfrak{p} - 1$ in place of $m_\mathfrak{p}$ because of the fixed determinant condition: $\det \rho = \phi_{\det}$, which kills the contribution of $\delta_{m_\mathfrak{p},\mathfrak{p}}$.

We now study $\Omega_{R^D_0/R^I_0} \otimes_{R^D_0} \mathbb{J}$ in (4.4). We look at R^D_∞ and R^I_∞. From the exact sequence:

$$1 \to I_\mathfrak{p} \to D_\mathfrak{p} \to (\mathrm{Frob}_\mathfrak{p})^{\widehat{\mathbb{Z}}} \to 1,$$

taking into account the fact that prime-to-p part of $(\mathrm{Frob}_\mathfrak{p})^{\widehat{\mathbb{Z}}}$ does not affect the base change calculation done in Corollary 3.2 (1), we see that the image in $R^D_{\mathfrak{p},0}$ of

$$R^I_{\mathfrak{p},\infty,\overline{\delta}_\mathfrak{p}} \otimes_{\mathcal{O}[[I^{ab}_{\mathfrak{p},\infty,p}]]} \mathcal{O}[[D^{ab}_{\mathfrak{p},\infty,p}]]$$

is isomorphic to

$$R^D_{\overline{\delta}_{i,\mathfrak{p}},F} = R^I_{\overline{\delta}_{i,\mathfrak{p}},F} \otimes_{\mathcal{O}[[I_{\mathfrak{p},ab}]]} \mathcal{O}[[D^{ab}_{\mathfrak{p},p}]],$$

where $I_{\mathfrak{p},ab}$ is the image of $I_\mathfrak{p}$ in $D^{ab}_{\mathfrak{p},p}$. This shows

$$\Omega_{R^D_{0,\overline{\delta}_{i,\mathfrak{p}}}/R^I_{0,\overline{\delta}_{i,\mathfrak{p}}}} \cong R^D_{0,\overline{\delta}_{i,\mathfrak{p}}} \otimes_{\mathbb{Z}_p} \Gamma.$$

Then we obtain from (4.4) the following exact sequence:

$$\bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}]^{m_{\mathfrak{p}}-1} \overset{\iota_D}{\to} \Omega_{R_F^\phi/R_0^I} \otimes \mathbb{J} \to \Omega_{R_F^\phi/R_0^D} \otimes \mathbb{J} \to 0. \tag{4.6}$$

4.3. SPECULATION ON THE ORDER OF THE TRIVIAL ZERO. Coming to this point, we would ask when the morphisms ι_I in (4.5) and ι_D in (4.6) are injective. If it is the case, the characteristic power series in $\mathbb{J}[[T]]$ of $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_\infty}$ would have trivial zero of order $\geq \sum_{\mathfrak{p}\in S_E}(m_{\mathfrak{p}}-1)|S_{\mathfrak{p}}|$ at $T=0$. On the other hand, heuristically, the order of such zero should be equal to the number of linear Euler p-factors vanishing at $s=0$ of the complex L-functions $L(s,\mathrm{Ad}_S(\varphi))$ if $\mathrm{Ad}_S(\varphi)$ is associated to a critical motive $\mathrm{Ad}_S(M)$ as in Example 2.7. More precisely, we need to count the linear factors, vanishing at $s=0$, of the modified p-Euler factor defined [H96b] Section 3,5 (E), but in our case of $\mathrm{Ad}_S(M)$, the two numbers match; so, we can use the number of such factors of the original p-Euler factor of the complex L-function.

To speculate about the order of the trivial zero, we return to the situation in Example 2.7. .Thus M is crystalline, $F=\mathbb{Q}$, the coefficients of M is also $\mathbb{Q}$, and φ is p-nearly ordinary of Borel-type. We write e_p for the number of linear factors in the Euler p-factor of $L(s,\mathrm{Ad}_{\mathrm{SL}(n)}(M))$ vanishing at $s=0$. Then e_p is the multiplicity of the eigenvalue 1 of the crystalline Frobenius acting on the p-adic crystalline realization $H_{crys}(\mathrm{Ad}(M))$, which is equal to $e_p=n-1$ and also to the $\mathbb{Z}_p$-rank of $H^0(D_p,\mathrm{gr}(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi)))$. Here the graded module $\mathrm{gr}(\mathrm{Ad}_{\mathrm{SL}(n)}(\varphi))$ is defined with respect to the flag: $(\mathrm{fil}_{\mathfrak{p}})$. The number $e_{\mathfrak{p}}$ can be defined for general φ (not necessarily associated to a crystalline motive) as follows:

$$e_{\mathfrak{p}} = \mathrm{rank}_{\mathbb{J}}\, H^0(D_{\mathfrak{p}},\mathrm{gr}(\mathrm{Ad}_S(\varphi))),$$

where $D_{\mathfrak{p}}$ is the decomposition subgroup at $\mathfrak{p}\in S_F$ of $\mathfrak{H}$. We see easily (from the argument proving Lemma 2.2) that under (RG_F) and ($\mathrm{Z}_{p,F}$)

$$e_{\mathfrak{p}} = m_{\mathfrak{p}} - 1. \tag{4.7}$$

When $\mathrm{Spec}(\mathbb{J})$ has densely populated points associated to critical motives $\mathrm{Ad}_S(M)$, φ is forced to have values in either a symplectic or orthogonal group G, because $\mathrm{Ad}_S(M)$ is not critical in the $\mathrm{GL}(n)$-case ($n>2$) (see Example 2.7 and Example 2.8). Since $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_\infty}\otimes_{\mathbb{J}[[T]]}\mathbb{J}\cong\Omega_{R_F^\phi/R_0^I}\otimes\mathbb{J}$ by Proposition 4.1, it might be natural to conjecture

CONJECTURE 4.2: *Suppose that*

(1) *The group S is isomorphic to either* $\mathrm{Sp}(n)$ *or* $\mathrm{SO}(n)$*;*

(2) $\mathbb{J} \in \mathrm{CNL}_{\mathcal{O}}$ *is an integral domain of characteristic zero;*

(3) $\varphi\colon \mathfrak{G} \to G(\mathbb{J})$ *is nearly ordinary of Borel-type;*

(4) $\mathrm{Ad}_S(\varphi)$ *is arithmetic in the following sense: for densely populated points* P *in* $\mathrm{Spec}(\mathbb{J})$, $\mathrm{Ad}_S(\varphi) \bmod P$ *is associated to a critical motive.*

Then we have the following two exact sequences:

$$0 \to \bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}]^{m_{\mathfrak{p}}-1} \to \Omega_{R_F^\phi/R_0^I} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F} \to 0,$$

$$0 \to \bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}]^{m_{\mathfrak{p}}-1} \to \Omega_{R_F^\phi/R_0^I} \otimes \mathbb{J} \to \Omega_{R_F^\phi/R_0^D} \otimes \mathbb{J} \to 0.$$

The first exact sequence is known to be true, when φ is modular two dimensional (see Corollary 5.4). We will prove as Proposition 7.1 the second exact sequence in many cases where again φ is modular two-dimensional.

Anyway we note the following consequence of our argument.

THEOREM 4.3: *Let F_∞/F be a $\mathbb{Z}_p$-extension satisfying* (TR). *Suppose that $\mathbb{J}$ is an integral domain of characteristic* 0. *If* $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_j}$ *is of $\mathbb{J}$-torsion for one* $j > 0$, *then* $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_\infty}$ *is a torsion module of finite type over* $\mathbb{J}[[T]] = \mathbb{J}[[\Gamma]]$ *for* $\Gamma = \mathrm{Gal}(F_\infty/F) = \gamma^{\mathbb{Z}_p}$ *with* $T = \gamma - 1$.

Proof: Let $t = \prod_\varepsilon(\gamma - \varepsilon(\gamma)) \in \mathcal{O}[[\Gamma]]$, where ε runs over all characters of Γ of order p^j. For $M = \bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}]^{m_{\mathfrak{p}}-1}$, M/tM is a torsion $\mathbb{J}$-module (which is killed by $\prod_\varepsilon(1 - \varepsilon(\gamma))$). Since $t | \gamma^{p^j} - 1$, we have the following exact sequence for $\mathrm{Sel}_j^* = \mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_j}$ from (4.5):

$$M/tM \to \mathrm{Sel}_\infty^* / t\,\mathrm{Sel}_\infty^* \to \mathrm{Sel}_j^* / t\,\mathrm{Sel}_j^* \to 0,$$

and we conclude that $\mathrm{Sel}_\infty^* / t\,\mathrm{Sel}_\infty^*$ is a torsion $\mathbb{J}$-module. This implies that Sel_∞^* is a torsion $\mathbb{J}[[T]]$-module, because t is a parameter of $\mathbb{J}[[T]]$ over $\mathbb{J}$. ∎

4.4. CONTROL OF TWISTED ADJOINT SELMER GROUPS. Now suppose that $p \nmid |\Delta| = [F : E]$ for $\Delta = \mathrm{Gal}(F/E)$ and $F_\infty = FE_\infty$ for a $\mathbb{Z}_p$-extension E_∞/E. We pick an absolutely irreducible representation $\overline{\psi}\colon \Delta \to \mathrm{GL}_m(\mathbb{F})$. We write $\psi\colon \Delta \to \mathrm{GL}_m(\mathcal{O}) \subset \mathrm{GL}_m(\mathbb{J})$ for the unique lift of $\overline{\psi}$, that is, $\psi \bmod \mathfrak{m}_{\mathbb{J}} = \overline{\psi}$. It is easy to see, under (RG_E) and $(\mathrm{Z}_{p,E})$,

$$e_{\mathfrak{p}}(\psi) = \mathrm{rank}_{\mathbb{J}} H^0(D_{\mathfrak{p}}, \mathrm{gr}(\mathrm{Ad}_S(\varphi)) \otimes \psi) = (m_{\mathfrak{p}} - 1) \dim_{\mathbb{F}} \mathrm{Hom}_\Delta(\mathbb{F}[S_{\mathfrak{p}}], \overline{\psi}). \tag{4.8}$$

If our speculation in the previous section is right, $e_p(\psi) = \sum_{\mathfrak{p}} e_{\mathfrak{p}}(\psi)$ should give the order of trivial zero of the characteristic power series in $\mathbb{J}[[T]]$ at $T = 0$ of

$\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)$. We prove in this section that we have an exact control of the Selmer group $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F_\infty}$ without error term if $\mathrm{Hom}_\Delta(\mathbb{F}[S_\mathfrak{p}],\overline{\psi})=0$ for all $\mathfrak{p}$; so, if $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F}$ is $\mathbb{J}$-torsion (as expected when $\mathrm{Ad}(\varphi)$ is arithmetic), $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F_\infty}$ is a torsion $\mathbb{J}[[T]]$-module without trivial zero at $T=0$. We will show this last statement in many cases when $G=\mathrm{GL}(2)$ in Section 6.

If we suppose that the ψ-isotypic component $\mathbb{J}[S_F][\psi]$ vanishes for $S_F=\sqcup_\mathfrak{p}S_\mathfrak{p}$, then we have from (4.5) and (4.6) that

$$\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))_{/F_j}[\psi]\cong\Omega_{R^\phi_{F_j}/R^I_j}\otimes\mathbb{J}[\psi]\cong\Omega_{R^\phi_{F_j}/R^D_j}\otimes\mathbb{J}[\psi]. \tag{4.9}$$

The above fact combined with (4.9) and Propositions 4.1 and 2.4 yields the desired result:

THEOREM 4.4: *Suppose that $\mathrm{Hom}_\Delta(\mathbb{F}[S_F],\overline{\psi})=0$ for an absolutely irreducible representation $\overline{\psi}$: $\Delta\to\mathrm{GL}_m(\mathbb{F})$. Suppose $p\nmid[F:E]$, and let ψ: $\Delta\to\mathrm{GL}_m(\mathcal{O})$ be the unique lift of $\overline{\psi}$. Let E_∞/E be a $\mathbb{Z}_p$-extension in which p totally ramifies. Let κ: $\mathfrak{G}\to\mathcal{O}[[\Gamma]]^\times$ be the tautological character inducing $\kappa:\mathrm{Gal}(E_\infty/E)\cong\Gamma$. Then, for a generator γ of $\Gamma=\mathrm{Gal}(F_\infty/F)$ for $F_\infty=FE_\infty$, we have*

$$\frac{\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi))\otimes\psi)_{/F_\infty}}{(\gamma-1)\,\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F_\infty}}\cong\frac{\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi\kappa)_{/F}}{(\gamma-1)\,\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi\kappa)_{/F}}$$
$$\cong\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F}.$$

In particular, $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi\kappa)_{/F}$ is of $\mathbb{J}[[\Gamma]]$-torsion if and only if $\mathrm{Sel}^*(\mathrm{Ad}_S(\varphi)\otimes\psi)_{/F}$ is of $\mathbb{J}$-torsion.

Example 4.1: Let ρ: $\mathfrak{G}\to\mathrm{GL}_2(\mathbb{J})$ be a Galois representation nearly ordinary of Borel type. Then we consider the symmetric k-th tensor representation $\varphi=\mathrm{Sym}^k(\rho)$: $\mathfrak{G}\to\mathrm{GL}_{k+1}(\mathbb{J})$, which is again of Borel type, and assume that φ satisfies (Z_F) and (RG_F). Then we have a decomposition valid over $\mathbb{J}$:

$$\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi)\cong\bigoplus_{j=1}^{k}\det(\rho)^{-j}\,\mathrm{Sym}^{2j}(\rho).$$

The component $\varphi_j=\det(\rho)^{-j}\,\mathrm{Sym}^{2j}(\rho)$ has absolutely irreducible reduction modulo $\mathfrak{m}_\mathbb{J}$ if for example $\mathrm{Im}(\rho\bmod\mathfrak{m}_\mathbb{J})\supset\mathrm{SL}_2(\mathbb{F})$ and $p>2k+1$. Since ρ is nearly ordinary, φ and hence each of φ_j: $\mathfrak{G}\to\mathrm{GL}_{2j+1}(\mathbb{J})$ is nearly ordinary. The Selmer datum of $\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi)$ induces that of φ_j, that is,

$$V^-_\mathfrak{p}(\varphi_j)=V^+_\mathfrak{p}(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))\bigcap V(\varphi_j),$$

and $V_{\mathfrak{p}}^{+}(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi)) = \bigoplus_{j=1}^{k} V_{\mathfrak{p}}^{-}(\varphi_j)$. Then we have the following isomorphism:

$$\mathrm{Sel}(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))_{/L} \cong \bigoplus_{j=1}^{k} \mathrm{Sel}(\varphi_j)_{/L}.$$

We now analyse how the module $\mathbb{J}[S_{\mathfrak{p}}]^k$ with trivial Γ-action in $\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))_{/F_j}$ is distributed to $\mathrm{Sel}^*(\varphi_j)_{/F_j}$. Fix a Levi-torus $T_{\mathfrak{p}}$ of $P_{\mathfrak{p}}$. We thus have the character $D_{\mathfrak{p}} \to T_{\mathfrak{p}}(\mathbb{J})$, which induces the morphism of $R_{G,L}^{D_{\mathfrak{p}}}$ into R_L^{ϕ}. We claim that the Lie algebra of this torus $\mathfrak{t}$ intersects with $V(\varphi_j)$, and the intersection is a rank 1 direct summand, that is, $\mathfrak{t} = \bigoplus_{j=1}^{k}(\mathfrak{t} \cap V(\varphi_j))$. This follows from the following fact: Let $t_{\mathfrak{p}}$ be the Levi-torus of the Borel subgroup $b_{\mathfrak{p}} \subset \mathrm{GL}(2)$ with $\mathrm{Im}(\rho|_{D_{\mathfrak{p}}}) \subset b_{\mathfrak{p}}(\mathbb{J})$. Then $t_{\mathfrak{p}} = \mathbf{G}_m \times \mathbf{G}_m$ is a 2-dimensional split torus in $\mathrm{GL}(2)_{/\mathbb{J}}$. The rational representation $V(\det^{-j} \otimes \mathrm{Sym}^{2j})$ is then decomposed into the direct sum of weight spaces $V(i)$ of $t_{\mathfrak{p}}$ for integers $-j \le i \le j$ on which $(t,s) \in t_{\mathfrak{p}} = \mathbf{G}_m \times \mathbf{G}_m$ acts via the character $(t,s) \mapsto t^i s^{-i}$. In particular, we have $\mathfrak{t} \cap V(\varphi_j) = V(0)$ and $V_{\mathfrak{p}}^{-}(\varphi_j) = \bigoplus_{i>0} V(i)$ (if we order the two factors $\mathbf{G}_m$ of $t_{\mathfrak{p}}$ properly). This shows our claim.

From this, we conclude one component $\mathbb{J}[S_{\mathfrak{p}}]$ is distributed to each $\mathrm{Sel}^*(\varphi_j)_{/F_j}$. We thus have the following exact sequences:

$$\bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}] \xrightarrow{\iota_I} \frac{\mathrm{Sel}^*(\varphi_j)_{/F_\infty}}{(\gamma-1)\,\mathrm{Sel}^*(\varphi_j)_{/F_\infty}} \to \mathrm{Sel}^*(\varphi_j)_{/F} \to 0, \tag{4.10}$$

$$\bigoplus_{\mathfrak{p}} \mathbb{J}[S_{\mathfrak{p}}] \xrightarrow{\iota_D} \mathrm{Sel}^*(\varphi_j)_{/F_j} \to \mathrm{Sel}^*_{st}(\varphi_j)_{/F_j} \to 0, \tag{4.11}$$

where $\mathrm{Sel}^*_{st}(\varphi_j)_{/F_j}$ is the strict Selmer group defined in 2.1. In particular, we get the following exact sequence:

$$\mathbb{J}^{e(\psi)} \to \frac{\mathrm{Sel}^*(\varphi_j \otimes \psi)_{/F_\infty}}{(\gamma-1)\,\mathrm{Sel}^*(\varphi_j \otimes \psi)_{/F_\infty}} \to \mathrm{Sel}^*(\varphi_j \otimes \psi)_{/F} \to 0, \tag{4.12}$$

where $e(\psi) = \dim_{\mathbb{F}} \mathrm{Hom}_{\Delta}(\mathbb{F}[S_F], \overline{\psi})$ and the Selmer group is defined with respect to $V_{\mathfrak{p}}^{-}(\varphi_j \otimes \psi) = V_{\mathfrak{p}}^{-}(\varphi_j) \otimes_{\mathcal{O}} V(\psi)$ for Artin representations $\psi \colon \Delta \to \mathrm{GL}_m(\mathcal{O})$.

Suppose that ρ is associated to a rank 2 pure motive $M_{/E}$ with critical $\mathrm{Ad}_{\mathrm{SL}(2)}(M)$. This is equivalent to assuming the following three conditions:

(1) ρ is associated to a rank 2 regular motive M;

(2) E is totally real;

(3) $\det(\rho)(c) = -1$ for all complex conjugation c.

Then φ_j is associated to a motive $M_j = \det(M^\vee)^j \otimes \mathrm{Sym}^{2j}(M)$. The motive M_j is critical if and only if j is odd. Thus the Conjecture 4.2 predicts the injectivity of ι_H for $H = I, D$ for every odd j (see Example 6.2).

5. Galois representations and p-adic Hecke algebras

To a cohomological Hecke eigen-form on a reductive group $H_{/F}$, we hope to associate a Galois representation into its Langlands dual $G = H^L$ over a sufficiently large $\mathcal{O}$. If $H = \mathrm{GL}(2)_{/F}$ for a totally real F, the association is known by the work of several number theorists, notably, Shimura, Wiles, Taylor and Blasius–Rogawsky. In this case, $G = \mathrm{GL}(2)_{/\mathcal{O}}$. When the representation $\overline{\rho}: \mathfrak{G} \to \mathrm{GL}_2(\mathbb{F})$ is modular over a totally real F in an appropriate sense (see below) and is nearly ordinary of Borel type, we can prove that the universal deformation ring R_F^{ϕ} is finite flat over an appropriate local Iwasawa algebra of the p-inertia group. The idea is due to Mazur that this ring should be isomorphic to an appropriate Hecke algebra for H. After carrying out this task of identifying the deformation ring with Hecke algebra (following the method of Wiles–Taylor and Fujiwara [Fu]), the finiteness and flatness follows from my earlier works on Hecke algebras [H88] and [H89a].

It might look odd to study Hilbert modular forms for the extension F rather than looking into modular forms for the base field E. Although a Δ-invariant 2-dimensional Galois representation attached to a Hilbert modular form for F (that is, a Galois representation modular over F) is expected to be modular over E, this is a hard question in Langlands theory (Galois descent or base-change in the automorphic side has not been fully established yet; see [HM]). However assuming modularity over a bigger field F (without assuming that over E), we can prove many fine results on the adjoint Selmer groups, which we are going to exhibit.

We keep the notation introduced in the earlier sections. In particular, we write $\mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E)$, $\mathfrak{H} = \mathrm{Gal}(F^{(p,\infty)}/F)$ and $\Delta = \mathrm{Gal}(F/E)$. We like to identify the various universal deformation rings with the corresponding Hecke algebras constructed out of Hilbert modular forms. We assume that $\phi_{\det} = \chi\mathcal{N}^m$ ($m \in \mathbb{Z}$) for the global p-adic cyclotomic character $\mathcal{N}$ taking the geometric Frobenius element $Frob_{\mathfrak{q}}$ at a prime $\mathfrak{q} \nmid p$ to $N(\mathfrak{q})$ and a finite order character $\chi: \mathfrak{H} \to \mathcal{O}^{\times}$. We assume F to be totally real throughout this section.

5.1. Properties of p-adic Hecke algebras. We shall define the Hecke algebra $h^?(p^\infty;\mathcal{O})_F$ corresponding to the functors $\Phi_F^? = \Phi_{\mathrm{GL}(2),F}^?$ for $? = n.ord$, ord and ϕ. Here we recall that $\rho: \mathfrak{H}_L \to \mathrm{GL}_2(A) \in \Phi_L^{n.ord}(A)$ is called **$\mathfrak{p}$-ordinary** if $\delta_{\mathfrak{p},1,\rho}$ is unramified. We call ρ p-ordinary if ρ is $\mathfrak{p}$-ordinary for all $\mathfrak{p} \in S_L$. We define Φ_L^{ord} to be the subfunctor of $\Phi_L^{n.ord}$ made of ordinary deformations. We put $\Phi_L^{\phi,ord}(A) = \Phi_L^{ord}(A) \bigcap \Phi_L^{\phi}(A)$. Under ($\mathrm{RG}_L$) and ($\mathrm{Z}_L$), $\Phi_L^?$ for $? = (\phi, ord)$ and ord is representable by a universal couple $(R_L^?, \varrho_L^?)$.

Let $O = O_F$ be the integer ring of F, and put $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{\mathfrak{p} \in S_F} O_{\mathfrak{p}}$. We write Z (resp. $C_{\infty+}$) for the center of $H = \mathrm{Res}_{O/\mathbb{Z}} \mathrm{GL}(2)$ (resp. the connected component of the standard maximal compact subgroup of $H(\mathbb{R})$). We consider for each open subgroup U of $Z(\mathbb{A}^{(\infty)})H(\widehat{\mathbb{Z}})$ the complex modular variety

$$X(U) = H(\mathbb{Q})_+ \backslash H(\mathbb{A})_+ / UZ(\mathbb{R})C_{\infty+},$$

where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$, $\mathbb{A} = \mathbb{A}^{(\infty)} \times \mathbb{R}$, $H(\mathbb{Q})_+ = H(\mathbb{Q}) \bigcap H(\mathbb{A})_+$ and $H(\mathbb{A})_+ = H(\mathbb{A}^{(\infty)})H(\mathbb{R})_+$ for the identity connected component $H(\mathbb{R})_+$ of the Lie group $H(\mathbb{R})$. When $U \supset Z(\mathbb{A}^{(\infty)})$, we may regard $X(U)$ as a modular variety of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}(2)$.

We define open compact subgroups of $H(\mathbb{A}^{(\infty)})$ for an ideal N of F by

$$U_0(N) = \left\{ x \in H(\widehat{\mathbb{Z}}) \middle| \, x \bmod N \in B(O/N) \right\}, \tag{5.1}$$

$$U_{11}(N) = \left\{ u \in H(\widehat{\mathbb{Z}}) \middle| \, u \bmod N \in U_B(O/N) \right\}, \tag{5.2}$$

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}, \tag{5.3}$$

where $\widehat{\mathbb{Z}} = \prod_{\ell:prime} \mathbb{Z}_\ell$, B is the standard upper triangular Borel subgroup of $\mathrm{GL}(2)_{/\mathbb{Z}}$, and U_B is the unipotent radical of B.

We assume that the quotient field K of $\mathcal{O}$ contains $\sigma(F) \subset \overline{\mathbb{Q}}_p$ for all field embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$. For each embedding σ: $F \hookrightarrow K$, we have the projection σ: $H(\mathbb{Q}_p) \to \mathrm{GL}_2(K)$. We consider the space of a polynomial representation of G

$$L(n, v; K) \cong \bigotimes_\sigma (\det(\sigma)^{v_\sigma} \otimes \mathrm{Sym}^{\otimes n_\sigma}(\sigma))$$

for the symmetric m-th tensor representation $\mathrm{Sym}^{\otimes m}(\sigma)$ of σ: $H(\mathbb{Q}_p) \to \mathrm{GL}_2(K)$. We regard n and v as linear combinations of embeddings of F into K with coefficients $n_\sigma \geq 0$ and v_σ. To make things more precise, we take $L(n, v; A)$ to be the space of polynomials in $\{(X_\sigma, Y_\sigma)\}_{\sigma \in I}$, with coefficients in an $\mathcal{O}$-module A, homogeneous of degree n_σ for each pair (X_σ, Y_σ). We let $\gamma \in U$ act on $P \in L(n, v; A)$ by

$$\gamma P((X_\sigma, Y_\sigma)) = \det(\gamma_p)^v P((X_\sigma, Y_\sigma)^t \sigma(\gamma_p)^\iota), \tag{5.4}$$

where $a^v = \prod_\sigma \sigma(a)^{v_\sigma}$.

By class field theory, we may regard the character χ as a Hecke character χ: $(F_{\mathbb{A}}^{(\infty)})^\times / F_+^\times \to \mathcal{O}^\times$, where $F_+^\times$ is the subgroup of totally positive elements in $F^\times$. Let $\mathfrak{c}$ be the conductor of χ_p in $O_p^\times = \prod_{\mathfrak{p}|p} O_{\mathfrak{p}}^\times$. When $U \subset U_1(\mathfrak{c})$, we define an action of $H(\mathbb{Q}) \times UZ(\mathbb{A})$ on $H(\mathbb{A})_+ \times L(n,v;K)$ by $\gamma(x,P)uz = (\gamma xuz, \chi(z)u_p^\iota P)$ for $u \in U$ with $u^\iota = \det(u)u^{-1}$, $\gamma \in H(\mathbb{Q})$ and $z \in Z(\mathbb{A})$. The module $L(n,v;A)$ with this extended action will be written as $L(n,v,\chi;A)$. Note that $U_0(\mathfrak{c}) \subset UZ(\mathbb{A}^{(\infty)})$. Then, for $U \subset U_0(\mathfrak{c})Z(\mathbb{A}^{(\infty)})$, we define the covering spaces $\mathcal{X}(U) \to X(U)$ by

$$\mathcal{X}(U) = H(\mathbb{Q})_+ \backslash (H(\mathbb{A})_+ \times L(n,v,\chi;A))/UC_{\infty+}. \tag{5.5}$$

Now we consider the sheaf of locally constant sections of $\mathcal{X}(U)$ over $X(U)$, which we write again as $L(n,v,\chi;A)$. For $d = [F:\mathbb{Q}]$, we consider

$$S(U;A) = H^d_{cusp}(X(U), L(n,v,\chi;A)). \tag{5.6}$$

we suppose that $S(U;K) \neq 0$ for a sufficiently small U.

Writing $L(A)$ for $L(n,v,\chi;A)$, we recall the definition of $H^d_{cusp}(X(U),L(A))$. When $n \neq 0$ or $d = [F:\mathbb{Q}]$ is odd, $H^d_{cusp}(X(U),L(\mathcal{O}))$ is defined to be the natural image of the compactly supported cohomology group $H^d_c(X(U),L(\mathcal{O}))$ in $H^d(X(U),L(K))$. When $n=0$ and $[F:\mathbb{Q}]$ is even, in $H^d(X(U),L(K))$, we have the space of invariant classes $\mathrm{Inv}(U)$ spanned by cohomology classes of the connected components of $X(U)$ (see [H88] Theorem 6.2). We then define

$$H^d_{cusp}(X(U),L(\mathcal{O})) = \mathrm{Im}\left(H^d_c(X(U),L(\mathcal{O})) \to \frac{H^d(X(U),L(K))}{\mathrm{Inv}(U)}\right).$$

Once $H^d_{cusp}(X(U),L(\mathcal{O}))$ is defined, we just put

$$H^d_{cusp}(X(U),L(A)) = H^d_{cusp}(X(U),L(\mathcal{O})) \otimes_{\mathcal{O}} A. \tag{5.7}$$

On this space, Hecke operators $T(y) = [U\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}U]$ naturally acts ([H88] Section 7); further, $\mathbb{T}(y) = y_p^{-v}T(y)$ preserves the $\mathcal{O}$-lattice $S(U;\mathcal{O})$. We consider the $\mathcal{O}$-subalgebra $h_{n,v}(U)$ of $\mathrm{End}_K(S(U;K))$ generated by $\mathbb{T}(y)$ for all integral ideles y, which is an algebra free of finite rank over $\mathcal{O}$.

We define a profinite group $\mathbf{G}$ by

$$\mathbf{G} = \mathrm{Cl}_F(p^\infty) \times O_p^\times, \tag{5.8}$$

where $\mathrm{Cl}_F(p^\infty) = \varprojlim_j \mathrm{Cl}_F(p^j)$ is the projective limit of the strict ray class groups $\mathrm{Cl}_F(p^j)$ of F modulo p^j. The central action $\langle z\rangle = z_p^{-n-2v}[UzU]$ of $z \in Z(\mathbb{A}^{(\infty)})$ is seen to be naturally contained in $h_{n,v}(U)$, and $h_{n,v}(U)$ is an algebra over $\mathcal{O}[[\mathbf{G}]]$ via the character

$$(z,y) \mapsto \langle z\rangle \mathbb{T}(y).$$

In particular, if $U = U_0(\mathfrak{c})$, (z, y) acts on $S(U; K)$ via the character $(z, y) \mapsto z^{-n-2v}y^{-v}$ although, if $n = v = 0$, the space is of weight 2 with the "Neben" character χ in classical sense.

We now gradually shrink U according to the deformation type ? introduced in the previous section to define the Hecke algebra associated to ?. There is a subgroup $W^? \subset \mathbf{G}$ associated to the deformation type "?". Thus we shall give a definition of the universal Hecke algebra of type W for closed subgroups $W \subset \mathbf{G}$. We suppose that $W = W^+ \times W^-$ with $W^- \subset O_p^\times$ and $W^+ \subset Cl_F(p^\infty)$. Writing $\pi\colon Z(\mathbb{A}^{(\infty)}) \cong (F_{\mathbb{A}}^{(\infty)})^\times \to Cl_F(p^\infty)$ for the projection, we put

$$(5.9) \qquad U_\alpha^W = \pi^{-1}(W^+)\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(p^\alpha) \middle|\, a_p \in (1 + p^\alpha O_p)W^-\right\}.$$

To each deformation type ?, we associate the following subgroup of $\mathbf{G}$, and define $\mathbf{G}^? = \mathbf{G}/W^?$:

$$(5.10) \qquad W^{\phi,ord} = \mathbf{G}, \quad W^{ord} = O_p^\times, \quad W^{n.ord} = \{1\}, \quad W^\phi = \mathrm{Cl}_F(p^\infty).$$

We look into the modular variety $X(U_\alpha^W)$. We see easily that $X(U_\alpha^{W^?}) = X(U_\alpha^?)$ for the following groups $U_\alpha^?$ which may look a bit different from $U_\alpha^{W^?}$ in appearance:

$$(5.11) \qquad U_\alpha^{ord} = U_1(p^\alpha), \quad U_\alpha^{n.ord} = U_{11}(p^\alpha), \quad \text{and} \quad U_\alpha^\phi = Z(\mathbb{A})U_\alpha^{n.ord}.$$

If we have several properties $P_1, \ldots, P_s$ giving deformation type, we define $W^{P_1,\ldots,P_s}$ by the subgroup of $\mathbf{G}$ generated by W^{P_j} for all j. The condition that $S(U_\alpha^W; K) \neq 0$ for sufficiently large α implies that $n + 2v = [n + 2v]\sum_{\sigma:F\hookrightarrow\overline{\mathbb{Q}}}\sigma$ for an integer $[n + 2v]$. Hereafter we assume that $m - 1 = [n + 2v]$ when "?" involves "$\phi = \chi\mathcal{N}^m$". If the property "?" does not involve "ϕ", $\kappa = (n, v)$: $(z, y) \mapsto N_{F/\mathbb{Q}}(z)^{-[n+2v]}y^{-v}$ factors through $\mathbf{G}^?$. If "?" involves "$\phi = \chi\mathcal{N}^m$", the character: $(z, y) \mapsto y^{-v}$ for $\kappa = (n, v)$ factors through $\mathbf{G}^?$, because in this case, we give the $\mathcal{O}[[\mathbf{G}^\phi]]$-algebra structure of the Hecke algebra $h(U_\alpha^\phi)$ by the character: $\mathbf{G}^\phi \ni y \mapsto \mathbb{T}(y)$.

We have a commutative diagram for $0 < \alpha < \beta$

$$\begin{array}{ccc} S(U_\alpha^W; K) & \longrightarrow & S(U_\beta^W; K) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ S(U_\alpha^W; K) & \longrightarrow & S(U_\beta^W; K) \end{array}$$

where $T = \mathbb{T}(y)$ and $\langle z\rangle$, respectively. Thus restriction of operators gives surjective $\mathcal{O}$-algebra homomorphisms: $h(U_\beta^W) \to h(U_\alpha^W)$. We then define

$$h(U_\infty^W; \mathcal{O})_F = \varprojlim_\alpha h(U_\alpha^W),$$

which is a compact local ring well defined independently of $\kappa = (n, v)$. Since the projection maps take $\mathbb{T}(y)$ to $\mathbb{T}(y)$, we have well defined $\mathbb{T}(y)$ in $h(U_\infty^W; \mathcal{O})$. In particular, for a prime element $\varpi_\mathfrak{q}$ of $F_\mathfrak{q}$ ($\mathfrak{q} \nmid p$), $\mathbb{T}(\varpi_\mathfrak{q})$ and $\langle \varpi_\mathfrak{q} \rangle$ are independent of the choice of $\varpi_\mathfrak{q}$, which we therefore write as $\mathbb{T}(\mathfrak{q})$ and $\langle \mathfrak{q} \rangle$.

Writing $h(U_\infty^W; \mathcal{O})_F = \prod_{h_F} h_F$ as a product of local rings h_F with maximal ideal m_h, we write $\mathbb{T}_h(y)$ for the projection of $\mathbb{T}(y)$ to h_F. Then we define the nearly p-ordinary part of $h(U_\infty^W; \mathcal{O})_F$ by

$$\text{(Hecke)} \qquad h^W(p^\infty; \mathcal{O})_F = \prod_{\mathbb{T}_h(p) \in h_F^\times} h_F.$$

We write in particular $h^?(p^\infty; \mathcal{O})_F$ for $h^W(p^\infty; \mathcal{O})_F$ if $W = W^?$.

For each character $\kappa = (n \geq 0, v, \varepsilon, \xi) : \mathbf{G} \to \mathcal{O}^\times$ given by

$$\langle z \rangle \mathbb{T}(y) \mapsto N(z)^{-[n+2v]} \varepsilon(z) y^{-v} \xi(y),$$

we consider the κ-eigen subspace $S(U_\alpha^{n.ord}; K)[\kappa]$ of $S(U_\alpha^{n.ord}; K)$ for sufficiently large α so that (ε, ξ) factors through $\mathrm{Cl}_F(p^\alpha) \times (O/p^\alpha)^\times$. Then we define the Hecke algebra $h_\kappa(\mathcal{O})$ by the $\mathcal{O}$-subalgebra of $\mathrm{End}_\mathcal{O}(eS(U_\alpha^{n.ord}; K)[\kappa])$ generated by $\mathbb{T}(y)$ and $\langle z \rangle$, where e is the idempotent of $h^{n.ord}(p^\infty; \mathcal{O})$ in $h(U_\infty^{n.ord}; \mathcal{O})$. Then this algebra is well determined only by κ and independent of α.

As shown in [H89a] Theorem 2.4 (see also Lemma 3.10), we have a canonical morphism for κ: $\mathbf{G}/W \to \mathcal{O}^\times$ as above

$$(\kappa) \qquad h^W(p^\infty, \mathcal{O}) \otimes_{\mathcal{O}[[\mathbf{G}]], \kappa} \mathcal{O} \twoheadrightarrow h_\kappa(\mathcal{O}),$$

which is a priori surjective and proven to be of finite kernel. Actually, we can now prove that (κ) is an isomorphism (for $p \geq 3$) purely in an automorphic way; see [H99c]. Here we are going to show the fact via the deformation theory of Galois representations. We write $\mathbb{G}_p^?$ (resp. $\mathbf{G}_p^?$) for the torsion-free part (resp. the maximal p-profinite part) of $\mathbf{G}^?$. From (κ), we conclude that the natural map

$$(\mathbf{G}^?) \qquad h^{n.ord}(p^\infty, \mathcal{O}) \otimes_{\mathcal{O}[[\mathbf{G}]], \pi_?} \mathcal{O}[[\mathbf{G}^?]] \twoheadrightarrow h^?(p^\infty; \mathcal{O})$$

is surjective and has a $\mathcal{O}[[\mathbb{G}_p^?]]$-torsion kernel of finite type. Here if ? involves "ϕ", the above tensor product over $\mathcal{O}[[\mathbf{G}]]$ is taken via the projection: $\mathcal{O}[[\mathbf{G}]] \to \mathcal{O}[[\mathbf{G}^?]]$ twisted by $\mathcal{N}^{m-1}$; for example, if $? = \chi\mathcal{N}^m$, the morphism π_ϕ sends $(z, y) \in \mathbf{G}$ to $N^{1-m}(z)y \in \mathcal{O}[[\mathbf{G}^\phi]]$. This is because we have given $\mathcal{O}[[\mathbf{G}^\phi]]$-algebra structure via the algebra homomorphism: $\mathcal{O}[[\mathbf{G}^\phi]] \to h^\phi$ taking $y \in \mathbf{G}^\phi$

to $\mathbb{T}(y)$. When "?" does not involve "ϕ", the morphism $\pi_?$ is induced by the projection $\mathbf{G} \to \mathbf{G}^?$.

Now we list several other properties of $h^?(p^\infty;\mathcal{O})$ we need later:

(ff) There exists an $\mathcal{O}[[\mathbf{G}_p^?]]$-free module $M^?$ of finite rank such that:

(1) $h^?(p^\infty;\mathcal{O})$ acts faithfully on M;

(2) For each arithmetic prime $P = \mathrm{Ker}(\kappa) \in \mathrm{Spec}(\mathcal{O}[[\mathbb{G}_p^?]])$, we have the local identity: $h^?(p^\infty;\mathcal{O})_P \cong M_P^?$ of $h^?(p^\infty;\mathcal{O})_P$-modules; in particular, $h^?(p^\infty;\mathcal{O})$ is a torsion-free $\mathcal{O}[\mathbb{G}_p^?]]$-module of finite type;

(3) $M^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}]],\pi_?} \mathcal{O}[[\mathbf{G}^?]] \cong M^?$.

The assertion (ff) follows from [H89a] Theorem 3.8, where we have shown the existence of a well controlled free $\mathcal{O}[[\mathbf{G}_p]]$-module $M^{n.ord}$ and $M^{\phi,ord}$ on which the Hecke algebra acts faithfully. When $F = \mathbb{Q}$, $M^{\phi,ord}$ is the $\pm$-eigenspace of complex conjugation in $eH^1_{cusp}(X_1^{ord},\mathcal{O})$ for the projector $e = \lim_{n\to\infty}\mathbb{T}(p)^{n!}$. When $F \neq \mathbb{Q}$, to construct $M^?$, we need to take a division quarternion algebra B over F unramified everywhere at finite places and most ramified at infinite places, and $M^{n.ord}$ is given either by $Y^{n.ord}$ in [H89a] page 167 or by the "+" or "−" eigenspace for complex conjugation of $Y^{n.ord}$ according as $[F:\mathbb{Q}]$ is even or odd. The module $M^{\phi,ord}$ is constructed also similarly to the case of $F = \mathbb{Q}$ using the modular variety $X_B(U)$ for B. The same construction applies to other cases, using $X_B(U^?_\infty) = \varprojlim{}_\alpha X_B(U^?_\alpha)$ in place of $X_B(U^{n.ord}_\infty)$ in [H89a].

A prime ideal P of a local ring $h^?$ of $h^?(p^\infty;\mathcal{O})$ is called **arithmetic** if $P \cap \mathcal{O}[[\mathbf{G}^?]]$ coincides with $\mathrm{Ker}(\kappa)$ for a $\kappa = (n,v,\varepsilon,\xi)$ with $n \geq 0$ ($\Leftrightarrow n_\sigma \geq 0$ for all $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$). For an arithmetic prime ideal P of $h^?$, we write $\mathcal{O}(P)$ for the p-adic integer ring of $k(P) = (h^?/P)\otimes_{\mathcal{O}} K$. Regarding P as a point P: $h^? \to \mathcal{O}(P)$ of $\mathrm{Spec}(h^?)$, we can associate to P a unique Hecke eigenvector $f_P \in S(U_\alpha^{n.ord};k(P))[\kappa]$ with $f|\mathbb{T}(y) = P(\mathbb{T}(y))f_P$ and a Galois representation ρ_P: $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathcal{O}(P))$ ([T] and [BR]; see also [H89b]). We call $\overline{\rho}$ **modular** (of level p) if there exists a local ring $h^{n.ord}$ of $h^{n.ord}(p^\infty;\mathcal{O})$ such that $\overline{\rho} \cong \rho_P \bmod \mathfrak{m}_{\mathcal{O}(P)}$ for an arithmetic point $P \in \mathrm{Spec}(h^{n.ord})$. Hereafter we assume $\overline{\rho}$ is modular of level p, and write $h^?$ for the local ring of $h^?(p^\infty;\mathcal{O})$ covered by $h^{n.ord}$ under the map $(\mathbf{G}^?)$. We assume the **absolute irreducibility** of $\overline{\rho}$ (AI_F). Under this condition, we may assume that ρ_P itself has values in $\mathrm{GL}_2(h^?/P)$, and then the isomorphism class of ρ_P over $h^?/P$ is unique (cf. Carayol [Cr]). Since the projector $e = \lim_{n\to\infty}\mathbb{T}(p)^{n!}$: $h(U^?_\infty;\mathcal{O}) \to h^?(p^\infty;\mathcal{O})$ kills the p-old part, the algebra $h^?$ is reduced if $\overline{\rho}$ satisfies (RG_F). We now list some properties satisfied by ρ_P.

(rep) We have a Galois representation $\rho_?$: $\mathfrak{H} \to \mathrm{GL}_2(h^?)$ which is $\mathfrak{m}_{h^?}$-adically continuous, and $\rho_?$ mod P is isomorphic to ρ_P over $h^?/P$ for all arithmetic points $P \in \mathrm{Spec}(h^?)$;

(NO_p) $\rho_?$ is nearly p-ordinary with the property "?" and satisfies $\delta_{1,\rho_?,\mathfrak{p}}([y;\mathfrak{p}]) = \mathbb{T}_h(y)$ for $y \in F_\mathfrak{p}$;

(Ch) $\det(1_2 - \rho_?(\mathrm{Frob}_\mathfrak{q})X) = 1 - \mathbb{T}(\mathfrak{q})X + \chi(\mathrm{Frob}_\mathfrak{q})N(\mathfrak{q})\langle\mathfrak{q}\rangle X^2 \in h^?[X]$ if $\mathfrak{q} \nmid p$.

5.2. Universality of Hecke algebras. Here, using Fujiwara's result as a seed, we identify $h^?_F$ with the universal ring $R^?_F$ under appropriate conditions. We suppose hereafter in this section that the order of χ is prime to p and $\phi = \chi\mathcal{N}$ (i.e. $m = 1$). Let $\kappa_0 \in \mathrm{Spec}(\mathcal{O}[[\mathbf{G}]])(\mathcal{O})$ which is induced by the trivial character. By our definition, κ_0 factors through $\mathbf{G}^?$. Then by (κ), $h^? \otimes_{\mathcal{O}[[\mathbf{G}]],\kappa} \mathcal{O}$ is an $\mathcal{O}$-module of finite type, and its maximal torsion-free quotient h^κ is canonically isomorphic to the local ring of $h_\kappa(\mathcal{O})$ corresponding to the deformation of $\overline{\rho}$. When $F = \mathbb{Q}$, h^{κ_0} is the direct local summand of the Hecke algebra of weight 2 with "Neben" character χ in the classical sense. We write $h^{\phi,ord}$ for h^{κ_0} because it should corresponds to $\Phi_F^{\phi,ord}$.

Let $I_{\mathfrak{p},ab}$ be the image of the inertia group $I_\mathfrak{p}$ in the maximal p-profinite abelian quotient $D^{ab}_{\mathfrak{p},p}$ of $D_\mathfrak{p}$. We put $I_F = \prod_{\mathfrak{p}\in S_F} I_{\mathfrak{p},ab}$. We have a character $\delta_\varrho = \prod_{\mathfrak{p}|p} \delta_{1,\varrho,\mathfrak{p}}\colon I_F \to R^?_F$, and hence $R^?_F$ is an $\mathcal{O}[[I_F]]$-algebra via δ_ϱ. By local class field theory, the inertia group I_F can be identified with the maximal p-profinite subgroup of $O_p^\times = \prod_{\mathfrak{p}|p} O_\mathfrak{p}^\times$. By global class field theory, we may regard $\det \varrho^?$ as a character of $\mathrm{Cl}_F(p^\infty)$ with values in $R^?_F$. Thus $R^?_F$ also has a natural algebra structure over $\mathcal{O}[[\mathbf{G}]]$.

Now we consider the following two conditions:

(NR_p) F is unramified at p;

(LD_p) F is linearly disjoint from $\mathbb{Q}(\mu_p)$ over $\mathbb{Q}$, and $F_p^\times$ for $F_p = F \otimes_\mathbb{Q} \mathbb{Q}_p$ is p-torsion-free.

Then the following result is shown by K. Fujiwara [Fu]:

Theorem 5.1 (K. Fujiwara): *Let p be an odd prime and $\chi\colon \mathfrak{H} \to \mathcal{O}^\times$ be a character of order prime to p. Put $\phi = \chi\mathcal{N}$. Suppose* (NR_p) *for F,* ($\mathrm{AI}_{F(\sqrt{(-1)^{(p-1)/2}p})}$), ($\mathrm{RG}_F$) *and that $\overline{\rho}$ is p-ordinary. Then we have*

(1) *The pair of the local ring $h^{\phi,ord} = h^{\kappa_0}$ and its Galois representation $\rho_{\kappa_0} = \rho_{\mathrm{Ker}(\kappa_0)}$ represents the functor $\Phi_F^{ord,\phi}$;*

(2) *$R_F^{\phi,ord} \cong h_F^{\phi,ord} = h^{\kappa_0}$ as $\mathcal{O}[[\mathbf{G}_p^{ord}]]$-algebras and $h^{\kappa_0} \cong M_\mathfrak{m}^{\phi,ord}$ as $h_F^{\phi,ord}$-modules, where $M^?_\mathfrak{m}$ is the localization of $M^?$ at the maximal ideal $\mathfrak{m}$ of $h^?_F$;*

(3) *The ring* $R_F^{\phi,ord}$ *is a local complete intersection. In other words,* $R_F^{\phi,ord} \cong \mathcal{O}[[T_1,\dots,T_r]]/(f_1,\dots,f_r)$ *for a regular sequence* $f_1,\dots,f_r$ *in a formal power series ring of* r *variables* $\mathcal{O}[[T_1,\dots,T_r]]$.

According to Fujiwara, we can replace the condition (NR_p) by the weaker condition (LD_p) if $\overline{\rho}$ is not flat at p. Irreducibility over $F(\sqrt{(-1)^{(p-1)/2}p})$ and linear disjointness of F from $\mathbb{Q}(\mu_p)$ are used to find primes $\mathfrak{q}$ outside the ramification of $\overline{\rho}$ such that $N(\mathfrak{q}) \equiv 1 \bmod p^m$ for any given integer $m > 0$ and $\overline{\rho}(\mathrm{Frob}_{\mathfrak{q}})$ has two distinct eigenvalues in $\mathbb{F}$, by the help of [DT] Lemma 3, which requires the surjectivity of the Teichmüller character modulo p. They are also used, as in [TW] Section 2, to find an auxiliary prime $\mathfrak{r}$ such that

(1) The local components of the Hecke algebra (associated to $\overline{\rho}$) of minimal level and of level $\mathfrak{r}$ added are isomorphic;
(2) The modular variety of level $\mathfrak{r}$ added is smooth, yielding torsion-free (or p-divisible by duality) cohomology group with coefficients in $\mathcal{O}$.

The use of auxiliary level $\mathfrak{r}$ should be removable, in the non-flat case, taking a sufficiently large p-power level (to assure smoothness instead of adding an auxiliary level $\mathfrak{r}$) and then returning to level p by taking $\mathbf{G}$-invariants (cf. [H89a] Lemma 3.10). If $\overline{\rho}$ is non-flat at $\mathfrak{p}$ and ordinary, the local condition: $\mu_p(F_{\mathfrak{p}}) = \{1\}$ can be also removed by using fixed determinant condition: $\det\rho = \phi$. If $\overline{\rho}$ is flat at $\mathfrak{p}|p$, we need to assume that $\mathfrak{p}$ is unramified over $\mathbb{Q}$, otherwise, the flat deformation problem is not well posed. Once the $\mathfrak{p}$-flat deformation problem is representable by a local component of the Hecke algebra of level $\mathfrak{p}$ removed (as proved by Fujiwara under the unramifiedness of $\mathfrak{p}$), one can proceed as Wiles [W] (3.11) to prove the universality of the corresponding local component of the Hecke algebra of $\mathfrak{p}$-included level for the deformation problem (ord,ϕ) (that is, the Selmer deformation in [W]). In this process, a lemma (due to Ribet; [W] Lemma 2.3) used by Wiles can be replaced by a similar one due to Taylor [T] Lemma 4 Case 1, when F has even degree over $\mathbb{Q}$. When F has odd degree over $\mathbb{Q}$, first take a totally real quadratic extension F'/F unramified at p, then prove the result over F', and finally descend to F as in [DHI]. Thus in the non-flat case, one gets the above result over the layers F_j. In the $\mathfrak{p}$-flat case, assuming $\mathfrak{p}$ is unramified in F, we have the above result at the bottom F (but not over the layers F_j). We should mention that Fujiwara's result actually covers Hecke algebras with (minimal) auxiliary level outside p. Anyway we now suppose the assertion of the above theorem and try to generalize it for other types "?" of deformation:

(univ) the pair of the local ring $h^{\kappa_0} = h^{\phi,ord}$ and its Galois representation $\rho_{\kappa_0} =$

$\rho_{\mathrm{Ker}(\kappa_0)}$ represents the functor $\Phi_F^{ord,\phi}$, and $h^{\phi,ord} \cong M_{\mathfrak{m}}^{\phi,ord}$ as $h_F^{\phi,ord}$-modules.

(cpi) the local ring $R_F^{\phi,ord}$ is a local complete intersection over $\mathcal{O}$.

Let ϕ' be any continuous arithmetic character ϕ': $\mathfrak{H} \to \mathcal{O}^\times$ with $\phi' \equiv \phi \bmod \mathfrak{m}_{\mathcal{O}}$. We suppose that ϕ' induces κ: $\mathbf{G}^{ord} = \mathrm{Cl}_F(p^\infty) \to \mathcal{O}^\times$, identifying $\mathrm{Cl}_F(p^\infty)$ with $\mathfrak{H}^{ab}$. For each $\rho \in \Phi_F^{\phi'}(A)$, ρ modulo the ideal $\mathfrak{a}$ of A generated by $\delta_\rho(\sigma) - 1$ for $\sigma \in I$ gives a p-ordinary deformation. Every p-ordinary deformation, which is a specialization of ρ, is a specialization of $\rho \bmod \mathfrak{a}$. Thus we get

$$\text{(Isom)} \qquad R_F^{\phi'} \otimes_{\mathcal{O}[[I_F]]} \mathcal{O} \cong R_F^{ord,\phi'},$$
$$R_F^{n.ord} \otimes_{\mathcal{O}[[I_F]]} \mathcal{O} \cong R_F^{ord} \quad \text{and} \quad R_F^{ord} \otimes_{\mathcal{O}[[\mathbf{G}_p^{ord}]],\kappa} \mathcal{O} \cong R_F^{ord,\phi'}.$$

Note that $h^{\phi'}$ is isomorphic to the subalgebra of $h^{n.ord}$ generated by $\mathrm{Tr}(\rho_{n.ord}^{\phi'})$, where $\rho_{n.ord}^{\phi'} = \rho_{n.ord} \otimes (\phi' \det(\rho_{n.ord})^{-1})^{1/2}$, without assuming (AI_F). This follows from the following fact: $M_{\mathfrak{m}}^{n.ord} = M_{\mathfrak{m}}^{\phi'} \widehat{\otimes} \mathcal{O}[[\mathbf{G}_p^{ord}]]$, where $\mathbf{G}_p^{ord}$ is the maximal p-profinite quotient of $Cl_F(p^\infty)$ (see [H99b] Chapter V Theorem 6.1, [H97a] Section 2 and [HM]). This shows $\rho_{n.ord}^{\phi'} \cong \rho_{\phi'}$.

Tensoring characters with representations can be performed both on the Galois side ($R_F^?$) and on the Hecke side ($h_F^?$), independently. Moreover, for each irreducible automorphic representation π of $H(\mathbb{A})$, $\pi \otimes \xi \cong \pi$ for a character ξ can happen only when π is an automorphic induction from a quadratic extension of F and ξ is the quadratic character associated to the quadratic extension ([L] Chapter 11 or [DHI] Lemma 3.2). This shows

$$\text{(TP)} \qquad h^{n.ord} \cong h^{\phi'} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathbf{G}_p^{ord}]] \quad \text{and} \quad h^{\phi'/\infty} \cong h^{\phi'} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\Gamma]]$$

on the Hecke side (cf. [H97a] Proposition 2.1 and [H99b] Chapter V Theorem 6.1). On the Galois side, the natural transformation $\Phi^{n.ord}(A) \ni \xi \mapsto (\xi^{\phi'}, \det(\xi)) \in \Phi^{\phi'}(A) \times \Phi_{\overline{\phi}}(A)$ induces

$$\text{(TP}'\text{)} \qquad R_F^{n.ord} \cong R_F^{\phi'} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathbf{G}_p^{ord}]] \quad \text{and} \quad R_F^{\phi'/\infty} \cong R_F^{\phi'} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\Gamma]].$$

We write $\mathfrak{a}_{I_F}$ for the augmentation ideal of $\mathcal{O}[[I_F]]$. We have the following commutative diagram with surjective arrows for the arithmetic character $\kappa_0 =$

(0, 0, id, id):

$$\begin{array}{ccc} \iota : R_F^{\phi} & \longrightarrow & h^{\phi} \\ \downarrow & & \downarrow \\ \iota' : R_F^{\phi,ord} = R_F^{\phi}/\mathfrak{a}_{I_F} R_F^{\phi} & \longrightarrow & h^{\phi}/\mathfrak{a}_{I_F} h^{\phi} \\ \wr\| \downarrow & & \downarrow \\ \iota'' : R_F^{\phi,ord} & \longrightarrow & h^{\kappa_0} = h_F^{\phi,ord} \cong M_{\mathfrak{m}}^{+}. \end{array}$$

We lift $1 \in h^{\kappa_0} \cong M_{\mathfrak{m}}^{\phi,ord} = M_{\mathfrak{m}}^{\phi}/\mathfrak{a}_{I_F} M_{\mathfrak{m}}^{\phi}$ to an element $m_0 \in M_{\mathfrak{m}}^{\phi}$. Consider the h^{ϕ}-linear map T: $h^{\phi} \to M_{\mathfrak{m}}^{\phi}$ given by $T(\theta) = \theta m_0$, which is surjective by Nakayama's lemma and is injective because h^{ϕ} acts faithfully on $M_{\mathfrak{m}}^{\phi}$. Thus $h^{\phi} \cong M_{\mathfrak{m}}^{\phi}$, which is $\mathcal{O}[[\mathbf{G}_p^{\phi}]]$-free of finite rank. Suppose that ι'' is an isomorphism ($\Leftrightarrow$ (univ)). This implies, again by Nakayama's lemma, ι has to be an isomorphism. Then by (TP) and (TP′), we further get $R_F^{n.ord} \cong h_F^{n.ord}$. By (Isom), we also have $R_F^{?} \cong h_F^{?}$ for ? = "*ord*", "(ϕ', ord)" and "ϕ'" for any arithmetic character ϕ': $\mathfrak{H} \to \mathcal{O}^{\times}$ with $\phi' \equiv \phi \bmod \mathfrak{m}_{\mathcal{O}}$. We can give a simpler proof of the fact: $R_F^{?} \cong h_F^{?}$ without using the modules $M_{\mathfrak{m}}^{?}$ but assuming that $F_p^{\times}$ is p-torsion-free (see [HM] Section 4). We record what we have proven:

THEOREM 5.2: *Let p be an odd prime. Let χ: $\mathfrak{H} \to \mathcal{O}^{\times}$ be the Teichmüller lift of a character with values in $\mathbb{F}^{\times}$. We put $\phi = \chi\mathcal{N}$. Let ϕ': $\mathfrak{H} \to \mathcal{O}^{\times}$ be an arithmetic character with $\phi' \equiv \phi \bmod \mathfrak{m}_{\mathcal{O}}$. Suppose that $\overline{\rho}$ is p-ordinary, modular of level p and satisfies the deformation property: "ϕ". If h^{κ_0} satisfies* (univ), *then for any combination ? of the properties: "n.ord", "ord" and "ϕ'", $(h^?, \rho_?)$ represents $\Phi_F^?$.*

COROLLARY 5.3: *Let the assumption and the notation be as in Theorem 5.2. Then*

(1) *$h^?$ is $\mathcal{O}[[\mathbf{G}_p^?]]$-free of finite rank. For every arithmetic character $\kappa = (n, v, \varepsilon, \xi)$ of $\mathbf{G}_p^?$ with $n \geq 0$, we have $h^? \otimes_{\mathcal{O}[[\mathbf{G}_p^?]],\kappa} \mathcal{O} \cong h^{\kappa}$.*
(2) *More generally the morphism*

$$(\mathbf{G}^?) \qquad h^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}_p]],\pi_?} \mathcal{O}[[\mathbf{G}_p^?]] \cong h^?$$

is a surjective isomorphism.

(3) *If $R_F^{\phi,ord} \cong h^{\kappa_0}$ is a local complete intersection over $\mathcal{O}$, then h^W is a local complete intersection over $\mathcal{O}[[\mathbf{G}_p/W]]$ for a subgroup $W \subset \mathbf{G}_p$ with torsion-free G_p/W, in other words, h^W is $\mathcal{O}[[\mathbf{G}_p/W]]$-free, and*

$$h^W \cong \mathcal{O}[[\mathbf{G}_p/W]][[T_1, \ldots, T_r]]/(f_1, \ldots, f_r)$$

for a regular sequence $f_1, \dots, f_r \in \mathcal{O}[[\mathbf{G}_p/W]][[T_1, \dots, T_r]]$.

(4) *If* $R_F^{\phi,ord} \cong h^{\kappa_0}$ *is a local complete intersection over* $\mathcal{O}$, *then* h^{κ} *is a local complete intersection over* $\mathcal{O}$ *for any character* $\kappa\colon \mathbf{G}_p \to \mathcal{O}$.

Proof: The assertions (1) and (2) follows from the proof of the theorem. We shall prove (3) and (4). By the assumption (cpi), we have an isomorphism

$$h^{\kappa_0} \cong \mathcal{O}[[T_1, \dots, T_r]]/(\overline{f}_1, \dots, \overline{f}_r).$$

We write $\overline{t}_j$ for the image of T_j in h^{κ_0}. Then we lift it to $t_j \in h^{n.ord}$ so that $t_j \otimes 1 = \overline{t}_j$ under $h^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}_p]]} \mathcal{O} \cong h^{\kappa_0}$. Then we define a surjective $\mathcal{O}[[\mathbf{G}_p]]$-linear map $\pi\colon \mathcal{O}[[\mathbf{G}_p]][[T_1, \dots, T_r]] \to h^{n.ord}$ by $T_j \mapsto t_j$. Then

$$\mathrm{Ker}(\pi) \otimes_{\mathcal{O}[[\mathbf{G}_p]]} \mathcal{O} \cong (\overline{f}_1, \dots, \overline{f}_r)$$

because $h^{n.ord}$ is $\mathcal{O}[[\mathbf{G}_p]]$-free. Then by the Nakayama's lemma, taking $f_i \in \mathrm{Ker}(\pi)$ so that $f_i \otimes 1 = \overline{f}_i$ under $\mathrm{Ker}(\pi) \otimes_{\mathcal{O}[[\mathbf{G}_p]]} \mathcal{O} \cong (\overline{f}_1, \dots, \overline{f}_r)$, we have $\mathrm{Ker}(\pi) = (f_1, \dots, f_r)$. Thus we see

$$h^{n.ord} \cong \mathcal{O}[[\mathbf{G}_p]][[T_1, \dots, T_r]]/(f_1, \dots, f_r).$$

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}[[\mathbf{G}_p]]$. By definition, we see

$$h^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}_p]]} \mathcal{O}[[\mathbf{G}_p]]/\mathfrak{a} \cong (\mathcal{O}[[\mathbf{G}_p]]/\mathfrak{a})[[T_1, \dots, T_r]]/(f_1^{\mathfrak{a}}, \dots, f_r^{\mathfrak{a}}),$$

where $f_j^{\mathfrak{a}}$ is the image of f_j in $(\mathcal{O}[[\mathbf{G}_p]]/\mathfrak{a})[[T_1, \dots, T_r]]$. If

$$\mathcal{O}[[\mathbf{G}_p]]/\mathfrak{a} \cong \mathcal{O}[[S_1, \dots, S_t, U_1, \dots, U_s]]/(g_1, \dots, g_s)$$

for a regular sequence $g_1, \dots, g_s$, we have

$$h^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}_p]]} \mathcal{O}[[\mathbf{G}_p]]/\mathfrak{a} \cong \frac{\mathcal{O}[[S_1, \dots, S_t, U_1, \dots, U_s, T_1, \dots, T_r]]}{(g_1, \dots, g_s, f_1^{\mathfrak{a}}, \dots, f_r^{\mathfrak{a}})}.$$

This shows the assertion (3) (resp. (4)) by taking $\mathfrak{a}$ to be the kernel of the projection: $\mathcal{O}[[\mathbf{G}_p]] \to \mathcal{O}[[\mathbf{G}_p/W]]$ (resp. $\kappa\colon \mathcal{O}[[\mathbf{G}_p]] \to \mathcal{O}$). ∎

Remark 5.1: For $\rho \in \Phi_F^{n.ord}(A)$, we define a character $\delta_\rho\colon I_F \to A^\times$ by $\prod_{\mathfrak{p}|p} \delta_{1,\rho,\mathfrak{p}}$. For each $\kappa = (n \geq 0, v, \varepsilon, \xi)$ as above, let $\phi_\kappa = \phi \mathcal{N}^{[n+2v]} \omega_p^{-[n+2v]} \varepsilon$, where ω_p is the Teichmüller character at p. Then we consider the following deformation functor of weight κ:

$$\Phi_\kappa(A) = \left\{ \rho \in \Phi_F^{\phi_\kappa}(A) \middle| \ \delta_\rho\colon I_F \to A^\times \text{ coincides with } [y, \mathfrak{p}] \mapsto y^{-v}\xi(y) \right\}.$$

In particular

$$\Phi_{\kappa_0}(A) = \left\{\rho \in \Phi_F^{\phi}(A) \middle|\ \rho \text{ is } p\text{-ordinary}\right\} = \Phi^{\phi,ord}(A).$$

Then it is easy to conclude from $h^{n.ord} \otimes_{\mathcal{O}[[\mathbf{G}_p]],\kappa} \mathcal{O} \cong h^{\kappa}$ that $(h^{\kappa}, \rho_{\kappa})$ represents the functor Φ_{κ} under the assumption of Theorem 5.2. Then the argument which proves Theorem 5.2 shows that

$$(h_F^{\phi_\kappa}, \rho_{\phi_\kappa}) \cong (R_F^{\phi_\kappa}, \varrho^{\phi_\kappa})$$

under the assumption of Theorem 5.2.

COROLLARY 5.4: *Let the notation and the assumption be as in the theorem. Let* $\mathrm{Spec}(\mathbb{J})$ *be a closed irreducible subscheme of* $\mathrm{Spec}(h^{n.ord})$ *of characteristic* 0, *and write* $\varphi\colon \mathfrak{H} \to \mathrm{GL}_2(\mathbb{J})$ *for the representation induced by* $\rho_{n.ord}$. *Suppose* (univ), (cpi) *and that that there exists an arithmetic point* $P \in \mathrm{Spec}(\mathbb{J})$. *Then* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ *is a torsion* $\mathbb{J}$*-module of homological dimension* 1, *and the following sequence is exact:*

$$0 \to \mathbb{J}[S_F] \to \Omega_{R_F^{\phi}/R_0^I} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} \to 0.$$

In other words, the first exact sequence of Conjecture 4.2 holds for φ.

Proof: We may assume that $\mathrm{Spec}(\mathbb{J}) \subset h^{\phi}$. We know $R^I_{\mathrm{GL}(2),F} \cong \mathcal{O}[[I_F]]$ and $R_0^I \cong \mathcal{O}[[I_0]]$ for the image I_0 of I_{F_∞} in I_F. By the exact sequence (4.3), we have the following exact sequence:

$$\mathbb{J}[S_F] \xrightarrow{\iota_I} \Omega_{R_F^{\phi}/R_0^I} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} \to 0.$$

Since $\mathcal{O}[[I_F]] = \mathcal{O}[[I_0]][[T_1, \ldots, T_s]]$ with $s = |S_F|$ and R_F^{ϕ} is free of finite rank over $\mathcal{O}[[I_F]]$, if ι_I is not injective, $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ cannot be a torsion $\mathbb{J}$-module.

Let us first deal with the case where $\mathbb{J} = \mathcal{O}$. Then $\mathrm{Spec}(\mathcal{O})$ is a closed subscheme of $\mathrm{Spec}(h^{\kappa})$ for the arithmetic character κ induced by $\mathcal{O}[[\mathbf{G}]] \hookrightarrow h^{\phi} \twoheadrightarrow \mathbb{J} = \mathcal{O}$. Since h^{κ} is reduced and free of finite rank over $\mathcal{O}$, $\Omega_{h^{\kappa}/\mathcal{O}}$ is a finite module. Thus by Theorem 2.3,

$$\begin{aligned}\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} &\cong \Omega_{h^{\phi}/\mathcal{O}[[I_F]]} \otimes_{h^{\phi}} \mathbb{J} \\ &\cong \Omega_{h^{\phi}/\mathcal{O}[[I_F]]} \otimes_{\mathcal{O}[[I_F]]} h^{\kappa} \otimes_{h^{\kappa}} \mathbb{J} \cong \Omega_{h^{\kappa}/\mathcal{O}} \otimes_{h^{\kappa}} \mathbb{J}\end{aligned}$$

is a finite module. This shows the injectivity of ι_I when $\mathbb{J} = \mathcal{O}$.

In general, specializing to $\mathbb{J}/P$, we get an isomorphism $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} \otimes_{\mathbb{J}} \mathbb{J}/P \cong \mathrm{Sel}^*(\mathrm{Ad}(\varphi \bmod P))_{/F}$, which is a finite module. Thus $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ is a torsion

$\mathbb{J}$-module, and the same argument as above proves the injectivity of ι_I for general $\mathbb{J}$.

Now we study the homological dimension of $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F}$ over $\mathbb{J}$. Here if $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is a minimal projective resolution of a $\mathbb{J}$-module M, the number n coincides with the homological dimension $\mathrm{hdim}_{\mathbb{J}} M$. Since

$$h^\phi \cong \mathcal{O}[[I_F]][[T_1, \ldots, T_r]]/(f_1, \ldots, f_r)$$

for a regular sequence $f_1, \ldots, f_r$, we have the following exact sequence:

$$\bigoplus_j h^\phi df_j \to \bigoplus_j h^\phi dT_j \to \Omega_{h^\phi/\mathcal{O}[[I_F]]} \to 0.$$

By tensoring $\mathbb{J}$ over h^ϕ, we get another exact sequence:

$$\bigoplus_j \mathbb{J} df_j \xrightarrow{\iota} \bigoplus_j \mathbb{J} dT_j \to \mathrm{Sel}^*(\mathrm{Ad}(\varphi)) \to 0.$$

Since $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))$ is a $\mathbb{J}$-torsion module, ι has to be injective, and thus we conclude

$$\mathrm{hdim}_{\mathbb{J}} \mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} = 1.$$

This finishes the proof. ∎

6. The order of the trivial zero

We continue to assume that F is totally real. We like to see the exact order of the zero at $T = 0$ of the characteristic power series of $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F_\infty}$ for a modular representation $\varphi\colon \mathfrak{H} = \mathrm{Gal}(F^{(p,\infty)}/F) \to \mathrm{GL}_2(\mathbb{J})$. We prove, under some conditions (including (univ)), the Selmer group $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F_\infty}$ is a torsion $\mathbb{J}[[T]]$-module of finite type with trivial zero at $T = 0$ of order $e_p = |S_F|$. For that, we need to assume a condition equivalent to the exactness of the second sequence of Conjecture 4.2, which is the non-vanishing of a certain jacobian determinant

$$\mathbf{J} = \det\left(\frac{\partial \mathbb{T}(\mathfrak{p}_i)}{\partial T_j}\right)$$

in $\mathbb{J}$ (Conjecture 6.2). This conjecture will be proven in Section 7 in some cases.

Suppose that φ extends to $\mathfrak{G} = \mathrm{Gal}(F^{(p,\infty)}/E) \to \mathrm{GL}_2(\mathbb{J})$ for a subfield E fixed by $\Delta \subset \mathrm{Aut}(F/\mathbb{Q})$. Then if $p \nmid [F : E]$, it follows from the above result and the consideration in 4.4 that $\mathrm{Sel}^*(\mathrm{Ad}(\varphi) \otimes \psi)_{/E_\infty}$ is torsion $\mathbb{J}[[T]]$-module with trivial zero at $T = 0$ of order $e_p(\psi) = \dim_{\mathbb{F}}(\mathrm{Hom}_\Delta(\mathbb{F}[S_F], \overline{\psi}))$ for any absolute irreducible representation $\psi\colon \Delta \to \mathrm{GL}_m(\mathcal{O})$ with $\overline{\psi} = \psi \bmod \mathfrak{m}_{\mathcal{O}}$.

We take a finite order character $\chi' \colon \mathfrak{H} \to \mathcal{O}^\times$ and put $\phi' = \chi' \mathcal{N}^m$ for an integer $m > 0$. We define a finite order character $\phi = \chi \mathcal{N}$ so that $\phi' \equiv \phi \bmod \mathfrak{m}_{\mathcal{O}}$ and $\chi \colon \mathcal{G} \to \mathcal{O}^\times$ is of order prime to p. By (TP′) in 5.2, we may assume $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(h^{\phi'})$ without losing generality. In what follows, F/E is always a Galois extension with Galois group Δ. We assume $p \nmid [F:E]$ throughout this section.

6.1. Another definition of Hecke algebras. A representation $\rho \colon \mathfrak{H} \to GL_2(\mathcal{O})$ with $\det(\rho) = \phi'$ is said to be **modular nearly ordinary** (of level p^∞) if (i) it is nearly ordinary of type $\{B_\mathfrak{p}\}$ for Borel subgroups $B_\mathfrak{p} \subset GL(2)$ and (ii) there exists a Hilbert modular form f such that

(1) f is a Hecke eigenform of p-power level and gives the representation ρ;

(2) $$f|\mathbb{T}(y) = \delta_\rho(([y,\mathfrak{p}])_{\mathfrak{p}|p}) f \text{ for all } y \in F_p^\times \text{ and all } \mathfrak{p}|p,$$

where $[y,\mathfrak{p}] = [y_\mathfrak{p},\mathfrak{p}]$ is the local Artin symbol and $\delta_\rho = \prod_\mathfrak{p} \delta_{1,\rho,\mathfrak{p}} \colon D_{ab} \to \mathcal{O}^\times$. Then we just make the maximal quotient $h = h_F$ ($\in \mathrm{CNL}_{\mathcal{O}}$) of $R_F^{\phi'}$ such that for all modular nearly ordinary deformations $\rho \colon \mathfrak{H} \to GL_2(\mathcal{O})$ of $\overline{\rho}$, the morphism $\iota \colon R_F^{\phi'} \to \mathcal{O}$ associated to ρ factors through h. By definition, h coincides with $h^{\phi'}$.

For each modular deformation ρ as above, one can associate a pure rank 2 motive $M_{\rho/F}$ with the etale realization $V(\rho)$ except for a few cases of weight 2 ([BR]). The Hodge type of $M_\rho \otimes_{F,\sigma} \mathbb{C}$ is given by

$$(n_\sigma + 1 + v_\sigma, v_\sigma),\ (v_\sigma, n_\sigma + 1 + v_\sigma) \quad \text{with } n_\sigma \geq 0.$$

By the fixed-determinant condition, $n_\sigma + 1 + 2v_\sigma = m$ for all $\sigma \colon F \hookrightarrow \overline{\mathbb{Q}}$. Therefore $v = \sum_\sigma v_\sigma \sigma$ determines n_σ, and $m - 2v_\sigma + 1$ is the classical weight of the Hilbert modular form f in (2) associated to ρ. By near-ordinarity, we see

$$\delta_\rho(([y,\mathfrak{p}])_{\mathfrak{p}|p}) = \xi(y) y^{-v} = \xi(y) \prod_{\sigma \colon F_p \to \overline{\mathbb{Q}}_p} (y^\sigma)^{-v_\sigma} \quad \text{for all } y \in O_p^\times = \prod_\mathfrak{p} O_\mathfrak{p}^\times$$

for a finite order character ξ. Then, under our terminology, the modular form f has weight $\kappa = (n \geq 0, v, \varepsilon, \xi)$, where $\phi' = \chi\varepsilon\mathcal{N}^m$ (i.e. $\chi'|_{I_F} = \chi\varepsilon$). We identify v with a character of I_F and then with a homomorphism of $\mathcal{O}$-algebra $v \colon \mathcal{O}[[I_F]] \to \mathcal{O} \in \mathrm{Spec}(\mathcal{O}[[I_F]])(\mathcal{O})$. We thus have $\phi' = \phi_\kappa$ for ϕ_κ as in Remark 5.1. Then by Remark 5.1, (h^κ, ρ_κ) represents Φ_κ under (univ). For any irreducible closed subscheme $\mathrm{Spec}(\mathbb{J})$ in $\mathrm{Spec}(R_F^{\phi'})$, a point $P \in \mathrm{Spec}(\mathbb{J})(\mathcal{O})$ is called **arithmetic** if it induces a character $y \mapsto \xi(y) y^v$ for a finite order character ξ and $v = \sum_\sigma v_\sigma \sigma$ with $v_\sigma \in \mathbb{Z}$. Thus if P is arithmetic, the Galois representation $\varphi_\mathbb{J} \bmod P$ is associated to a classical Hilbert Hecke eigenform and a pure rank 2 motive.

6.2. CONTROL AND TORSION THEOREMS FOR THE ADJOINT SELMER GROUPS. We take a $\mathbb{Z}_p$-extension F_∞/F satisfying (TR) as in Section 4 and use the notation introduced there. We recall that $D_L = \prod_{\mathfrak{p}\in S_L} D^{ab}_{\mathfrak{p},p}$, where $D^{ab}_{\mathfrak{p},p}$ is the maximal p-profinite abelian quotient of the decomposition group $D_\mathfrak{p}$ at $\mathfrak{p}$ of $\mathfrak{H}_L$. We write $I_L = \prod_{\mathfrak{p}\in S_L} I_{\mathfrak{p},ab}$ for the inertia subgroup $I_{\mathfrak{p},ab}$ of $D^{ab}_{\mathfrak{p},p}$. We define I_j (resp. D_j) by the image of I_{F_∞} (resp. D_{F_∞}) in I_{F_j} (resp. D_{F_j}). By local class field theory, I_j is isomorphic to a p-profinite subgroup of $O^\times_{j,p} = (O_{F_j}\otimes_\mathbb{Z}\mathbb{Z}_p)^\times$ which is made of universal norms from $F_{\infty,p}$. In other words,

$$I_{F_j}/I_j \cong \Gamma_j^{S_F} \quad\text{and}\quad D_{F_j}/D_j \cong \Gamma_j^{S_F}$$

for $\Gamma_j = \mathrm{Gal}(F_\infty/F)$ under total ramification of F_∞/F at p: (TR). Thus $\mathcal{O}[[I_F]] = \mathcal{O}[[I_0]][[T_1,\ldots,T_s]]$ with $s = |S_F|$ for parameters $T_1,\ldots,T_s$.

Note here that $R^I_j \cong \mathcal{O}[[I_j]]$. Thus we have from Theorem 2.3 that

$$\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J}))_{/F_\infty} \cong \Omega_{R^{\phi'}_{F_\infty}/\mathcal{O}[[I_\infty]]} \otimes_{R^{\phi'}_{F_\infty}} \mathbb{J}.$$

Here we note that $\mathcal{O}[[I_\infty]]$ is a gigantic non-noetherian ring close to $\mathcal{O}[[(\oplus_\mathfrak{p}\mathbb{Z}_p[[\Gamma_{0,\mathfrak{p}}]])]]$; nevertheless, we have already shown in Theorem 4.3 (see below) that $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J}))_{/F_\infty}$ is a torsion $\mathbb{J}[[\Gamma]]$-module of finite type if $\mathrm{Spec}(\mathbb{J})$ contains an arithmetic point, supposing (univ) for F_j for all $j<\infty$. Therefore $R^{\phi'}_{F_\infty}$ is miraculously close to $\mathcal{O}[[I_\infty]]$. Here is a direct consequence of Theorem 4.3:

THEOREM 6.1: *Suppose* (AI_F)*,* (RG_F) *and* (univ) *for* F. *Let* $\mathrm{Spec}(\mathbb{J})$ *be a closed irreducible subscheme of* $\mathrm{Spec}(h^{\phi'}_F)$ *containing an arithmetic point.*

1. *If* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J}))_{/F} = 0$, *then we have* $h^{\phi'}_F \cong \mathcal{O}[[I_F]]$.
2. *Suppose* (univ) *for* F_j *for one* $0<j<\infty$. *Then* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J}))_{/F_\infty}$ *is a torsion* $\mathbb{J}[[\Gamma]]$*-module of finite type.*
3. *Suppose that* φ_F *extends to* $\varphi\colon \mathfrak{G}\to GL_2(\mathbb{J})$ *and* $\mathrm{Hom}_\Delta(\mathbb{F}[S_F],\overline{\psi}) = 0$ *for an absolutely irreducible Artin representation* $\psi\colon \Delta\to GL_m(\mathcal{O})$. *Then if* $F_\infty = E_\infty F$ *for a* $\mathbb{Z}_p$*-extension* E_∞/E,

$$\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J})\otimes\psi)_{/E_\infty} = 0 \iff \mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J})\otimes\psi)_{/E} = 0,$$

and $\mathrm{Sel}^*(\mathrm{Ad}(\varphi)\otimes\psi)_{/E_\infty}$ *is a torsion* $\mathbb{J}[[T]]$*-module of finite type. Moreover if* (cpi) *holds for* F *and* $\mathbb{J}$ *is a regular local ring,*

$$\mathrm{hdim}_{\mathbb{J}[[T]]}\mathrm{Sel}^*(\mathrm{Ad}(\varphi)\otimes\psi)_{/E_\infty} = 1,$$

and $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_\mathbb{J})\otimes\psi)_{/E_\infty}$ *has no pseudo-null* $\mathbb{J}[[T]]$*-submodule non-null.*

Proof: If $\mathrm{Sel}(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F} = 0$, then $\Omega_{h^{\phi'}/\mathcal{O}[[I_F]]} \otimes \mathbb{J} = 0$ and by Nakayama's lemma, $\Omega_{h^{\phi'}/\mathcal{O}[[I_F]]} = 0$ and hence $R_F^{\phi} \cong h_F^{\phi'} \cong \mathcal{O}[[I_F]]$.

The torsion-ness of $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty}$ (the assertion (2)) follows from Theorem 4.3 and the $\mathbb{J}$-torsion of $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_j}$ for all finite j (see Corollary 5.4).

As for (3), under the disjointness of $\mathbb{F}[S_F]$ and $\overline{\psi}$, $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}) \otimes \psi)_{/F_\infty}$ is well controlled (see Theorem 4.4). Thus

$$\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}) \otimes \psi)_{/F_\infty} = 0 \iff \mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}) \otimes \psi)_{/F} = 0.$$

Write $M = \mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}) \otimes \psi)_{/F_\infty}$, and suppose $M \neq 0$. Then by Proposition 2.4 and Corollary 5.4, M is a torsion $\mathbb{J}[[T]]$-module of finite type. We know again from Proposition 2.4 and Corollary 5.4 that $\mathrm{hdim}_{\mathbb{J}} M/TM = 1$. Then by a theorem of Auslander and Buchsbaum ([M] Theorem 19.1),

$$\mathrm{hdim}_{\mathbb{J}} M/TM + \mathrm{depth}_{\mathbb{J}} M/TM = \mathrm{depth}_{\mathbb{J}} \mathbb{J}.$$

Since $\mathbb{J}$ is regular, $\mathbb{J}[[T]]$ is regular, and hence by a theorem of J.-P. Serre, $\mathrm{hdim}_{\mathbb{J}[[T]]} M < \infty$ (see [M] Theorem 19.2). Again by a theorem of Auslander and Buchsbaum, we see

$$\mathrm{hdim}_{\mathbb{J}[[T]]} M + \mathrm{depth}_{\mathbb{J}[[T]]} M = \mathrm{depth}_{\mathbb{J}[[T]]} \mathbb{J}[[T]] = \mathrm{depth}_{\mathbb{J}} \mathbb{J} + 1.$$

Since M/TM is a torsion $\mathbb{J}$-module, $\mathrm{depth}_{\mathbb{J}[[T]]} M = \mathrm{depth}_{\mathbb{J}} M/TM + 1$. This shows that

$$\mathrm{hdim}_{\mathbb{J}[[T]]} M = \mathrm{hdim}_{\mathbb{J}} M/TM = 1,$$

and we are done. ∎

To get a control result when $\mathrm{Hom}_\Delta(\mathbb{F}[S_F], \overline{\psi}) \neq 0$, we need to assume the exactness of the second exact sequence in Conjecture 4.2, since we know the exactness of the first. We like to interpret this exactness in terms of Hecke operators. We number the prime ideals over p of F as $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$. As already remarked, $I_F/I_0 \cong \Gamma^{S_F}$. Thus we can choose a set of parameters $T_1, \ldots, T_s \in \mathcal{O}[[I_F]]$ so that $\mathcal{O}[[I_F]] = \mathcal{O}[[I_0]][[T_1, \ldots, T_s]]$. We take a uniformizer $\{\varpi_i\}$ of $F_{\mathfrak{p}_i}$ which is a universal norm from $F_{\infty,\mathfrak{p}_i} = \bigcup_j F_{j,\mathfrak{p}_i}$. Let $\mathrm{Spec}(\mathbb{I})$ be an irreducible component of $\mathrm{Spec}(h_0)$ (thus $\mathbb{I}$ is a torsion-free $\mathcal{O}[[I_{F_0}]]$-module of finite type) containing $\mathrm{Spec}(\mathbb{J})$. Then we write $t_{\mathfrak{p}_i}$ for the image of $\mathbb{T}(\varpi_i)$ in $\mathbb{I}$. The element $t_{\mathfrak{p}_i}$ is unique up to multiplication by $I_0 \subset \mathcal{O}[[I_0]]^\times \subset \mathbb{I}^\times$. Since $dt_{\mathfrak{p}_i} \in \Omega_{\mathbb{I}/\mathcal{O}[[I_0]]}$ is the image of $\mathfrak{p}_i \in \mathbb{I}[S_F]$ under the map: $\mathbb{I}[S_F] \to \Omega_{h_F^{\phi'}/\mathcal{O}[[I_0]]} \otimes \mathbb{I}$ in Conjecture 4.2, for any element $\eta \in \mathbb{I}$ which kills the $\mathbb{I}$-torsion module $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{I}}))_{/F}$, $\eta dt_{\mathfrak{p}_i}$

is a linear combination of dT_j by the exactness of (4.5). Thus

$$\eta^s \det\left(\frac{\partial t_{\mathfrak{p}_i}}{\partial T_j}\right) \in \mathbb{I}.$$

Since we have from Theorem 2.3 that, for $R = R_F^{\phi'}$,

$$\begin{aligned}\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F} &\cong \Omega_{R/\mathcal{O}[[I_F]]} \otimes_R \mathbb{J} \\ &\cong (\Omega_{R/\mathcal{O}[[I_F]]} \otimes_R \mathbb{I}) \otimes_{\mathbb{I}} \mathbb{J} \cong \mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{I}}))_{/F} \otimes_{\mathbb{I}} \mathbb{J},\end{aligned}$$

if $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F}$ is a $\mathbb{J}$-torsion module (which is true under (univ) and the existence of an arithmetic point in $\mathrm{Spec}(\mathbb{J})$), we can find η as above with $\pi(\eta) \neq 0$ in $\mathbb{J}$. Thus we can think of

$$J = \pi\left(\det\left(\frac{\partial t_{\mathfrak{p}_i}}{\partial T_j}\right)\right)$$

in the field of fractions of $\mathbb{J}$ under these assumptions.

CONJECTURE 6.2: *Suppose* (AI_F) *and* (RG_F) *for* $\overline{\rho}_F = \varphi_{\mathbb{J}} \bmod \mathfrak{m}_{\mathbb{J}}$ *and that* $\mathrm{Spec}(\mathbb{J})$ *contains an arithmetic point. Then*

$$J = \pi\left(\det\left(\frac{\partial t_{\mathfrak{p}_i}}{\partial T_j}\right)\right)$$

does not vanish in the field of fractions of $\mathbb{J}$, *where* $\pi\colon \mathbb{I} \twoheadrightarrow \mathbb{J}$ *is the projection map.*

Note that the conjecture for $\mathbb{J} = \mathbb{I}$ is equivalent to the analytic independence of $\{t_{\mathfrak{p}_i}\}$ over $\mathcal{O}[[I_0]]$. Since $t_{\mathfrak{p}_i} = \delta_{1,\varphi,\mathfrak{p}_i}(\mathrm{Frob})$ for the Frobenius element $\mathrm{Frob} \in \mathfrak{H}_{F_\infty}$ at $\mathfrak{p}_i$, the image under the natural map: $\mathbb{J}[S_F] \to \Omega_{R_F^{\phi'}/\mathcal{O}[[D_0]]} \otimes \mathbb{J}$ of $\mathfrak{p}_i \in \mathbb{J}[S_F]$ is $dt_{\mathfrak{p}_i}$. Thus the above conjecture is equivalent to the exactness of the second sequence in Conjecture 4.2.

THEOREM 6.3: *Let the notation and the assumption be as in the conjecture. Suppose* (univ) *for* $j = 0$ *and* $J \neq 0$ *and let* $s = |S_F|$. *Then we have*

(1) *If* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F} = 0$ *and* $J \in \mathbb{J}^\times$, *then*

$$R_{F_j}^{\phi'} \cong R_{F_j}^{\phi'} \cong \mathcal{O}[[D_j]], \quad R_{F_j}^{n.ord} \cong h_{F_j}^{n.ord} \cong \mathcal{O}[[D_j \times Cl_{F_j}(p^\infty)_p]] \\ \text{and} \quad \mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty} \cong \mathbb{J}[S_F],$$

where $\mathrm{Cl}_{F_j}(p^\infty)_p$ *is the Galois group of the maximal* p*-profinite abelian extension unramified outside* p *and* ∞ *over* F_j. *In particular,* (univ) *holds for all* $j > 0$.

(2) $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty}$ *is a torsion* $\mathbb{J}[[\Gamma]]$*-module pseudo-isomorphic to* $\mathbb{J}^s \times M$, *where*

$$M = \Omega_{R^{\phi'}_\infty/\mathcal{O}[[D_\infty]]} \otimes_{R^{\phi'}_\infty} \mathbb{J}.$$

Moreover M/TM *is a torsion* $\mathbb{J}$*-module of finite type.*

(3) *Suppose that* $\mathbb{J}$ *is normal. Write* $\Phi(T) \subset \mathbb{J}[[T]]$ *(resp.* $\Psi(T)$*) for the characteristic ideal of* M *(resp.* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty}$*). Then we have*

$$\Psi(T) = \Phi(T)T^s, \quad \Phi(0) \neq 0 \quad \text{and} \quad \Phi(0)|J\eta,$$

where η *is the characteristic ideal of the* $\mathbb{J}$*-module* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F}$. *We have made use of the convention that* $\Phi(0) = (\Phi(T) + (T)/(T))$.

(4) *In addition to the assumption of* (3), *we further assume* (cpi) *and that* $\mathbb{J}$ *is a regular local ring. Then the characteristic ideals in* (3) *are all principal, and we write* $\Phi(T)$, $\Psi(T)$ *and* η *for the generators of the characteristic ideals. Then* $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty}$ *is a* $\mathbb{J}[[T]]$*-torsion module of homological dimension* 1 *and has no pseudo-null submodule non-null. Moreover, we have* $\Phi(0) = J\eta$ *up to units.*

Proof: We have two exact sequences:

$$0 \to \mathbb{J}[S_F] \xrightarrow{\iota_I} \Omega_{R^\phi_F/\mathcal{O}[[I_0]]} \otimes \mathbb{J} \to \mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F} \to 0, \tag{6.1}$$

$$0 \to \mathbb{J}[S_F] \xrightarrow{\iota_D} \Omega_{R^{\phi'}_{F_j}/\mathcal{O}[[I_j]]} \otimes \mathbb{J} \to \Omega_{R^{\phi'}_{F_j}/R^D_j} \otimes \mathbb{J} \to 0. \tag{6.2}$$

From the assumption of (1) and the above sequences, we conclude

$$\mathbb{J}[S_F] \cong \Omega_{R^\phi_F/\mathcal{O}[[I_0]]} \otimes \mathbb{J} \quad \text{and} \quad \Omega_{R^{\phi'}_{F_j}/R^D_j} \otimes \mathbb{J} = 0,$$

because $J \in \mathbb{J}^\times$ implies that $\mathrm{Im}(\iota_I) = \mathrm{Im}(\iota_D)$. By (4.2) and Nakayama's lemma, we see $\Omega_{R^{\phi'}_{F_j}/\mathcal{O}[[D_j]]} = 0$ and thus $R^{\phi'}_{F_j} \cong h^{\phi'}_{F_j} \cong \mathcal{O}[[D_j]]$. This shows that

$$\mathrm{Sel}^*(\mathrm{Ad}(\varphi_{\mathbb{J}}))_{/F_\infty} \cong \Omega_{\mathcal{O}[[D_\infty]]/\mathcal{O}[[I_\infty]]} \otimes \mathbb{J} \cong \mathbb{J}[S_F].$$

The identity $R^{n.ord}_{F_j} \cong h^{n.ord}_{F_j} \cong \mathcal{O}[[D_j \times \mathrm{Cl}_{F_j}(p^\infty)_p]]$ follows from (TP) and (TP′). This shows (1).

We now prove (2). We may assume that $t_{\mathfrak{p}_1}, \dots, t_{\mathfrak{p}_s} \in \mathcal{O}[[I_F]]$ inside $\mathbb{I}$, since the argument with suitable modification works well without this condition replacing $t_{\mathfrak{p}_j}$ by analytically independent $t'_{\mathfrak{p}_j}$ in $\mathcal{O}[[I_F]] \cap \mathcal{O}[[t_{\mathfrak{p}_1}, \dots, t_{\mathfrak{p}_s}]] \subset \mathbb{I}$. Write R_j for $R^{\phi'}_{F_j}$, A_j for $\mathcal{O}[[D_j]]$, Λ for $\mathcal{O}[[I_F]]$ and M for $\Omega_{R_\infty/A_\infty} \otimes_{R_\infty} \mathbb{J}$. Let

$$J_j = \mathrm{Ker}(R_j \to \mathbb{I}).$$

We have the following commutative diagram with exact rows and columns:

$$
(6.3)\qquad
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{J} & = & \Omega_{A_j/\mathcal{O}[[I_j]]} \otimes_{A_j} \mathbb{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} & & \\
(J_j/J_j^2) \otimes_{\mathbb{I}} \mathbb{J} & \longrightarrow & \Omega_{R_j/\mathcal{O}[[I_j]]} \otimes_{R_j} \mathbb{J} & \xrightarrow{b} & \Omega_{\mathbb{I}/\mathcal{O}[[I_j]]} \otimes_{\mathbb{I}} \mathbb{J} & \longrightarrow & 0 \\
\wr\Vert\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
(J_j/J_j^2) \otimes_{\mathbb{I}} \mathbb{J} & \longrightarrow & \Omega_{R_j/A_j} \otimes_{R_j} \mathbb{J} & \xrightarrow{d} & \Omega_{\mathbb{I}/A_j} \otimes_{\mathbb{I}} \mathbb{J} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0. & &
\end{array}
$$

Since $J \neq 0$ implies injectivity of f and e, from this sequence for $j = \infty$, we have the following exact sequence:

$$0 \to \Omega_{R_\infty/\mathcal{O}[[I_\infty]]} \otimes \mathbb{J} \xrightarrow{\beta} M \times (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{J}) \xrightarrow{\alpha} \Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{J} \to 0,$$

where $\alpha(m,a) = d(m) - h(a)$ and $\beta(a) = (g(a), b(a))$. Since $\Omega_{\mathbb{I}/A_0} \otimes_{\mathbb{I}} \mathbb{J}$ is $\mathbb{J}$-torsion by $J \neq 0$, it is $\mathbb{J}[[T]]$-pseudo-null. Again by $J \neq 0$, $\iota_D(\mathbb{J}[S_F]) \backslash (\Omega_{\mathbb{I}/\mathcal{O}[[I_0]]} \otimes_{\mathbb{I}} \mathbb{J})$ is $\mathbb{J}$-torsion and hence is $\mathbb{J}[[T]]$-pseudo-null. Since

$$\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F_\infty} \cong \Omega_{R_\infty/\mathcal{O}[[I_\infty]]} \otimes \mathbb{J},$$

this shows the assertion. By (4.2) applied to $H = D$, we have $M/TM \cong \Omega_{R_0/A_0} \otimes_{R_\infty} \mathbb{J}$. The latter differential module is a $\mathbb{J}$-torsion module, because the two exact sequences of Conjecture 4.2 are both exact by our assumption. Thus M is a $\mathbb{J}[[T]]$-torsion module, and hence $\mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F_\infty}$ is a torsion $\mathbb{J}[[T]]$-module of finite type.

By (2), $\Psi(T) = \Phi(T)T^s$ and $\Phi(0) \neq 0$. To see (3), we only need to prove $\Phi(0) | J\eta$. Since $\mathcal{O}[[t_{\mathfrak{p}_1}, \ldots, t_{\mathfrak{p}_s}]] \subset \mathcal{O}[[I_F]]$ inside $\mathbb{I}$ as we may assume, we see ι_D brings $\mathbb{J}[S_F]$ inside $\mathrm{Im}(\iota_I)$. Then $N = \mathrm{Coker}(\iota_D : \mathbb{J}[S_F] \hookrightarrow \mathrm{Im}(\iota_I))$ has homological dimension ≤ 1 and is $\mathbb{J}$-torsion, and the characteristic ideal of N is generated by J. On the other hand, we have the following exact sequence:

$$0 \to N \to \Omega_{R_0/\mathcal{O}[[D_0]]} \otimes \mathbb{J} \to \Omega_{R_0/\mathcal{O}[[I_F]]} \otimes \mathbb{J} \to 0$$

from the first exact sequence of Conjecture 4.2, which holds in our case (Corollary 5.4). Thus the characteristic ideal of $M/TM \cong \Omega_{R_0/\mathcal{O}[[D_0]]} \otimes \mathbb{J}$ is given by $J\eta$, and M/TM has homological dimension ≤ 1. In particular, $\mathrm{depth}_{\mathbb{J}[[T]]} M = \mathrm{depth}_{\mathbb{J}} M/TM + 1$. Anyway we have $\Phi(0) | J\eta$.

If we further suppose that $\mathbb{J}$ is regular, then $\mathrm{hdim}_{\mathbb{J}[[T]]} M < \infty$, and therefore,

$$\mathrm{hdim}_{\mathbb{J}[[T]]} M + \mathrm{depth}_{\mathbb{J}[[T]]} M = \mathrm{depth}_{\mathbb{J}[[T]]} \mathbb{J}[[T]] = \mathrm{depth}_{\mathbb{J}} \mathbb{J} + 1 \quad \text{and}$$
$$\mathrm{hdim}_{\mathbb{J}} M/TM + \mathrm{depth}_{\mathbb{J}} M/TM = \mathrm{depth}_{\mathbb{J}} \mathbb{J}.$$

Since we already know that $\mathrm{depth}_{\mathbb{J}[[T]]} M = \mathrm{depth}_{\mathbb{J}} M/TM+1$ and $\mathrm{hdim}_{\mathbb{J}} M/TM \leq 1$, we conclude $\mathrm{hdim}_{\mathbb{J}[[T]]} M \leq 1$. Then $\Phi(0) = J\eta$ because M has no pseudo-null module non-null. By the exact sequence (6.2) for $j = \infty$, we conclude

$$\mathrm{hdim}_{\mathbb{J}[[T]]} \mathrm{Sel}^*(\mathrm{Ad}(\varphi))_{/F_\infty} = 1,$$

which finishes the proof. ∎

The following result follows directly from the above theorem and Proposition 2.4:

COROLLARY 6.4: *Let the notation and the assumption be as in the theorem. Suppose that $F_\infty = E_\infty F$ for a $\mathbb{Z}_p$-extension E_∞/E and φ_F has an extension $\varphi_E\colon \mathfrak{G} \to \mathrm{GL}_2(\mathbb{J})$. Then writing $e(\psi)$ for the dimension of $\mathrm{Hom}_\Delta(\mathbb{F}[S_F], \overline{\psi})$ over $\mathbb{F}$ for an absolutely irreducible Artin representation $\psi\colon \Delta = \mathrm{Gal}(F/E) \to \mathrm{GL}_m(\mathcal{O})$, the characteristic ideal of $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_E) \otimes \psi)_{/E_\infty}$ has trivial zero of order $e(\psi)$ at $T = 0$. If $\mathbb{J}$ is regular and* (cpi) *holds for F, $\mathrm{Sel}^*(\mathrm{Ad}(\varphi_E)\otimes\psi)_{/E_\infty}$ has homological dimension* 1 *over $\mathbb{J}[[T]]$.*

Example 6.1: Let $F = \mathbb{Q}$ and $F_\infty = \mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension. Suppose that $\overline{\rho}$ is associated to a p-ordinary Hecke eigen-form $f \in S_k^{ord}(\Gamma_0(p))$ ($k \geq 2$). The condition $\Omega_{h^{ord}/\mathcal{O}[[I_{\mathbb{Q}}]]} = 0$ ($\iff h^{ord}_{/\mathbb{Q}} = \mathcal{O}[[\Gamma]]$) is equivalent to the fact that there is no congruence between f and any other Hecke eigenforms in $S_k^{ord}(\Gamma_0(p))(\cong S_k^{ord}(\mathrm{SL}_2(\mathbb{Z}))$ if $k > 2$). This condition: $\Omega_{h^{ord}/\mathcal{O}[[I_{\mathbb{Q}}]]} = 0$ is satisfied for $p = 11$ and $f(z) = \Delta(z) - \beta\Delta(pz) \in S_{12}(\Gamma_0(p))$, where $\Delta = \sum_{n=1}^{\infty} \tau(n)q^n$ is the Ramanujan's function $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and β is p-adic non-unit root of $X^2 - \tau(p)X + p^{11} = 0$. Thus identifying h^{ord} with $\mathcal{O}[[\Gamma]] \cong \mathcal{O}[[W]]$ for the weight variable $W = \gamma - 1$, we can regard $T(p)$ as a power series $a(W)$. Let $\mathbb{J} = h^{ord} = \mathcal{O}[[W]]$. Then $a(\gamma^{11} - 1)$ is the coefficient of f in q^p, and

$$J = \frac{\partial a}{\partial W} \equiv \frac{a(\gamma - 1) - a(\gamma^{11} - 1)}{p} \bmod \mathfrak{m}_{\mathbb{J}}.$$

On the other hand, for this f, we have

$$a(\gamma - 1) = 1 \text{ and } a(\gamma^{11} - 1) \equiv \tau(11) = 534612 \bmod 11^{11},$$

and $a(p;\Delta) - 1 = 534611 = 7 \cdot 11 \cdot 53 \cdot 131$. Y. Maeda* has checked this non-divisibility by p^2 of $a(\gamma - 1) - a(\gamma^p - 1)$ for $p = 17, 19, 23, 29, 31, 37, 41, 43, 59, 61$ and 67. Thus for these primes, $J \in \mathcal{O}[[W]]^\times$, and the assumption of the assertion (1) of Theorem 6.3 is satisfied. As for $p = 53$, there are two families of modular forms, and one of them with $a \equiv -1 \mod m_{\mathcal{O}[[W]]}$ has non-unit

$$J = \frac{\partial a}{\partial W}(W).$$

In this case, for the cusp form $f \in S_{54}(\Gamma_0(53))$ of weight 54 in the family, $a(p; f) + 1$ is divisible by a square of a prime factor of 53 in the Hecke field of f. Thus in this case, J is a non-unit, but $J \neq 0$ by Proposition 7.1 (see also Remark 7.1).

We now study a bit what we can say about the symmetric power of 2-dimensional representations.

Example 6.2: Let $\rho: \mathfrak{H} \to \mathrm{GL}_2(\mathbb{J})$ be a deformation of $\overline{\rho}$ for an irreducible closed subscheme $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(R)$ for $R = R^{\phi}_{\mathrm{GL}(2),F}$ of characteristic 0. We suppose (AI_F), (RG_F) and (univ) for $\overline{\rho}$ and that

$$\overline{\varphi} = \mathrm{Sym}^k(\overline{\rho}) \text{ satisfies } (\mathrm{Z}_F) \text{ and } (\mathrm{RG}_F). \tag{6.4}$$

For a positive integer k, we now put $\varphi = \mathrm{Sym}^k(\rho): \mathfrak{H} \to \mathrm{GL}_{k+1}(\mathbb{J})$ and decompose $\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi) = \bigoplus_{j=1}^k \varphi_j$ for $\varphi_j = \det(\rho)^{-j} \otimes \mathrm{Sym}^{2j}(\rho)$.

We consider the morphism of algebraic groups s: $\mathrm{SL}(2) \to \mathrm{SL}(k+1)$ induced by symmetric k tensors. Then its differential ds: $V(\mathrm{Ad}(\rho)) = \mathfrak{sl}(2) \to \mathfrak{sl}(k+1) = V(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))$ induces the inclusion of $V(\varphi_1)$ into $V(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))$ (which is unique up to scalar multiple). This map induces

$$ds_* : \mathrm{Sel}(\mathrm{Ad}_{\mathrm{SL}(2)}(\rho)) = \mathrm{Sel}(\varphi_1) \hookrightarrow \mathrm{Sel}(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi)).$$

We can reformulate the above argument as follows: Write (R', ϱ') (resp. (R, ϱ)) for the universal couple associated to $\Phi' = \Phi^{\phi^{k(k+1)/2}}_{\mathrm{GL}(k+1),L}$ (resp. $\Phi^{\phi}_{\mathrm{GL}(2),L}$) deforming $\overline{\rho} = \varphi \bmod \mathfrak{m}_{\mathbb{J}}$ (resp. $\rho \bmod \mathfrak{m}_{\mathbb{J}}$). Since $\mathrm{Sym}^k(\varrho) \in \Phi'(R)$, we have a unique morphism α: $R' \twoheadrightarrow R$ such that $\alpha\varrho' \approx \mathrm{Sym}^k(\varrho)$. This induces the following commutative diagram:

(6.5)

$$\begin{array}{ccccc}
\mathbb{J}[S_F]^k & \xrightarrow{\iota'} & \Omega_{R'/\mathcal{O}[[I_0^k]]} \otimes_{R'} \mathbb{J} \cong \bigoplus_{j=1}^k \frac{\mathrm{Sel}^*(\varphi_j)_{/F_\infty}}{T\,\mathrm{Sel}^*(\varphi_j)_{/F_\infty}} & \xrightarrow{\pi'} & \bigoplus_{j=1}^k \mathrm{Sel}^*(\varphi_j)_{/F} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{J}[S_F] & \xrightarrow[\iota]{} & \Omega_{R/\mathcal{O}[[I_0]]} \otimes_R \mathbb{J} \cong \frac{\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\rho))_{/F_\infty}}{T\,\mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\rho))_{/F_\infty}} & \xrightarrow[\pi]{} & \mathrm{Sel}^*(\mathrm{Ad}_{\mathrm{SL}(2)}(\rho))_{/F},
\end{array}$$

* The author is grateful to Y. Maeda for supplying the above data.

and the dual α^* of α gives an injection of $\mathrm{Sel}(\mathrm{Ad}_{\mathrm{SL}(2)}(\rho))_{/L}$ into $\mathrm{Sel}(\mathrm{Ad}_{\mathrm{SL}(k+1)}(\varphi))_{/L}$ such that $\alpha^* = ds_*$.

We have a similar result for $\mathrm{Ad}_{\mathrm{Sp}(k+1)}(\varphi) = \bigoplus_{j=0}^{\ell} \varphi_{2j+1}$ when $k = 2\ell+1$ and $\mathrm{Ad}_{\mathrm{SO}(k+1)}(\varphi) = \bigoplus_{j=0}^{\ell} \varphi_{2j+1}$ for $k = 2\ell+2$. Thus Conjecture 4.2 tells us that the p-adic L-function $L_p(s, \varphi_j)$ associated to $\mathrm{Sel}^*(\varphi_j)_{/F_\infty}$ for odd j to have a trivial zero of order $|S_F|$ at $s = 0$, but it gives no information on $L_p(s, \varphi_j)$ for even j.

7. Proof of Conjecture 6.2 in the case of multiplicative reduction

Define $W_{\mathfrak{p}} \subset O_{\mathfrak{p}}^\times$ by

$$W_{\mathfrak{p}} = \left\{ y \in O_{\mathfrak{p}}^\times \,\middle|\, \text{The local Artin symbol } [y, \mathfrak{p}] \text{ is trivial on } F_{\infty,\mathfrak{p}} \right\}.$$

For each subset $\Sigma \subset S_F$, writing $S = S_F = \Sigma \sqcup \Sigma^c$, we put

$$W_\Sigma = \mathrm{Cl}_F(p^\infty) \times \Big(\prod_{\mathfrak{p}\in\Sigma} W_{\mathfrak{p}} \Big) \times \Big(\prod_{\mathfrak{p}\in\Sigma^c} O_{\mathfrak{p}}^\times \Big) \subset \mathbf{G}.$$

Let $\mathrm{Spec}(\mathbb{I})$ be an irreducible component of $\mathrm{Spec}(h_F^\phi)$ and $\mathrm{Spec}(\mathbb{J})$ be the irreducible component of $\mathrm{Spec}(h^{W_\Sigma}(p^\infty;\mathcal{O}))$ contained in $\mathrm{Spec}(\mathbb{I})$. Here $h^W(p^\infty;\mathcal{O})$ is the Hecke algebra defined in 5.1.

Note that $\mathbf{G}/W_\Sigma \cong \prod_{\mathfrak{p}\in\Sigma} \boldsymbol{\Gamma}_{\mathfrak{p}}$ for the inertia subgroup $\boldsymbol{\Gamma}_{\mathfrak{p}}$ of Γ at $\mathfrak{p}$. By (TR), actually $\boldsymbol{\Gamma}_{\mathfrak{p}} = \Gamma$.

We order the prime factors of p in F as $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$.

PROPOSITION 7.1: *Let $\chi\colon \mathfrak{H} \to \mathcal{O}^\times$ be a finite order character, and put $\phi = \chi\mathcal{N}$. Let the notation be as in Conjecture 6.2. Suppose that F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension. Suppose that at least one arithmetic point $P\colon \mathbb{J} \to \mathcal{O}$ is associated to a p-divisible group of (potentially) multiplicative type at $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}$. Then $\{t_i \mid i = 1, \ldots, s\}$ in $\mathbb{J}$ are analytically independent over $\mathcal{O}$, and hence $J \neq 0$ in $\mathbb{J}$, where $t_i = \mathbb{T}(\varpi_i) - \omega(\mathbb{T}(\varpi_i))$ for the Teichmüller character $\omega\colon \mathbb{J}^\times \to \mu_{q-1}(\mathbb{J})$ with $q = \#(\mathbb{F})$.*

Proof: We put $S_j = \{\mathfrak{p}_{j+1}, \mathfrak{p}_{j+2}, \ldots, \mathfrak{p}_s\}$ for $j = 0, \ldots, s-1$. We write W_j for W_{S_j} and put $\mathbf{G}_j = \mathbf{G}/W_j \cong \prod_{k>j} \boldsymbol{\Gamma}_{\mathfrak{p}_k}$. We write h_j for $h^{W_j}(p^\infty;\mathcal{O})$ and let $\mathrm{Spec}(\mathbb{J}_j)$ be the (unique) irreducible component of $\mathrm{Spec}(h^{W_j}(p^\infty;\mathcal{O}))$ inside $\mathrm{Spec}(\mathbb{I})$. We have $\dim_{\mathcal{O}} \mathbb{J}_j = s - j$ by construction. Thus we get the following sequence of surjective algebra homomorphisms (or a stratification of $\mathrm{Spec}(\mathbb{J})$):

$$\mathbb{J} = \mathbb{J}_0 \xrightarrow{\pi_1} \mathbb{J}_1 \xrightarrow{\pi_2} \cdots \to \mathbb{J}_{s-1} \xrightarrow{\pi_s} \mathcal{O} \tag{7.1}$$

and $P = \pi_s \circ \pi_{s-1} \circ \cdots \circ \pi_1\colon \mathbb{J} \twoheadrightarrow \mathcal{O}$, because F_∞/F is cyclotomic.

By the multiplicative reduction assumption, $P(\mathbb{T}(\varpi_i))$ is a root of unity for $i < s$. Thus replacing $\mathbb{T}(\varpi)$ and t_i by $\mathbb{T}(\varpi_i)^h$ and $\mathbb{T}(\varpi_i)^h - \omega(\mathbb{T}(\varpi_i)^h)$ for a suitable integer exponent $h > 0$, we may assume that $P(t_i) = 0$ for all $i < s$. Then what we need to prove is:

(7.2) The elements $\{t_i | \, i = 1, \dots, s\}$ in $\mathbb{J}$ are analytically independent over $\mathcal{O}$.

For each $1 \le j \le s-1$, we take a quarternion algebra $B_{j/F}$ such that

(1) $B_j \otimes_F F_{\mathfrak{p}_i}$ for $1 \le i \le j$ is a division algebra;

(2) $B_j \otimes_F F_{\mathfrak{q}} \cong M_2(F_{\mathfrak{q}})$ for all prime ideals $\mathfrak{q}$ outside $\{\mathfrak{p}_i | \, i \le j\}$;

(3) $B_j \otimes_{F,\sigma} \mathbb{R}$ is a division algebra at as many embeddings $\sigma\colon F \hookrightarrow \mathbb{R}$ as possible.

For any arithmetic point Q in $\mathrm{Spec}(\mathbb{J}_j)$, the algebra homomorphism $Q\colon \mathbb{J}_j \to \overline{\mathbb{Q}}_p$ is associated to a classical Hilbert modular form f_Q with $f_Q | \mathbb{T}(\mathfrak{q}) = Q(\mathbb{T}(\mathfrak{q})) f_Q$. By our construction, the local component at $\mathfrak{p}_i$ for $i \le j$ of the automorphic representation spanned by f_Q is a Steinberg representation, which is a image under the Jacquet–Langlands–Shimizu correspondence of an automorphic representation of the algebraic group $B^\times_{j/F}$. Thus $Q(t_i) = 0$ for $i \le j$. Since arithmetic points are dense in $\mathrm{Spec}(\mathbb{J}_j)$, we conclude $\pi_j(t_i) = 0$ for $i \le j$:

$$\begin{array}{ccccccccccc}
\mathbb{J}_0 & \longrightarrow & \mathbb{J}_1 & \longrightarrow & \mathbb{J}_2 & \longrightarrow & \cdots \to & \mathbb{J}_{s-2} & \longrightarrow & \mathbb{J}_{s-1} \\
t_s & \longrightarrow & t_s & \longrightarrow & t_s & \longrightarrow & \cdots \to & t_s & \longrightarrow & t_s \\
t_{s-1} & \longrightarrow & t_{s-1} & \longrightarrow & t_{s-1} & \longrightarrow & \cdots \to & t_{s-1} & \longrightarrow & 0 \\
t_2 & \longrightarrow & t_2 & \longrightarrow & 0 & \longrightarrow & \cdots \to & 0 & \longrightarrow & 0 \\
t_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \to & 0 & \longrightarrow & 0.
\end{array}$$

On the other hand, t_j is transcendent over $\mathcal{O}$ in $\mathbb{I}_{j-1}$ because $|Q(\mathbb{T}(\varpi_j))| = |\varpi_j|_p^{-n_\sigma - 1}$ (the modified Ramanujan bound) for all arithmetic points Q of weight $\kappa = (n, v)$ factoring through $\mathbb{J}_j$, where $\sigma\colon F \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ induces $\mathfrak{p}_j$. This can be shown as follows: The classical Ramanujan bound is given by

$$|Q(T(\varpi_j))| = |\varpi_j|_p^{-n_\sigma - 2v_\sigma - 1}.$$

By definition, we have $\mathbb{T}(\varpi_j) = \varpi_j^{-v_\sigma} T(\varpi_j)$. Since F_∞/F is cyclotomic, we know that ϖ_j can be chosen in $F \subset F_{\mathfrak{p}_j}$ and $|\varpi_j^{-v_\sigma}| = |\varpi_j|_p^{2v_\sigma}$. This shows the identity $|Q(\mathbb{T}(\varpi_j))| = |\varpi_j|_p^{-n_\sigma - 1}$. Again using the fact that F_∞/F is cyclotomic, we know that there are infinitely many arithmetic points factoring through $\mathbb{J}_j$ with distinct n_σ. Thus by induction, we see t_j has infinitely many distinct values on the closed subscheme defined by the equation $t_1 = t_2 = \cdots = t_{j-1} = 0$, and hence t_j is analytically independent over $\mathcal{O}[[t_1, t_2, \dots, t_{j-1}]]$, which shows the result. ∎

Remark 7.1: By the result of [BDGP], if $\mathrm{Spec}(h^{n.ord}(p^\infty;\mathcal{O})_F)$ has an arithmetic point P associated to an elliptic curve E defined over $\mathbb{Q}$ with multiplicative reduction at p and if F has only one prime $\mathfrak{p}$ over p, then with the notation in Conjecture 6.2,

$$P(J) = P\Big(\frac{d\mathbb{T}(\varpi_{\mathfrak{p}})}{dT_1}\Big)$$

(see [GS] Section 2.4) is transcendental over $\mathbb{Q}$ and hence $P(J) \neq 0$. Thus if there is only one prime factor of p in F and $E_{/\mathbb{Q}}$ is minimally modular over F associated to an irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(h(p^\infty;\mathcal{O})_F)$, $J \neq 0$ holds for the projection $\pi\colon \mathbb{I} \to \mathbb{J}$ as long as $P \in \mathrm{Spec}(\mathbb{J})$ for closed irreducible $\mathrm{Spec}(\mathbb{J}) \subset \mathrm{Spec}(\mathbb{I})$.

8. Correction to [H96a]

The assertions (2) and (3) of Theorem 6.3 for $F = \mathbb{Q}$ and p-ordinary φ was first claimed in [H96a] using a method different from the one employed here. However, the proof given in [H96a] contains a gap stemming from a mis-statement of the assertion of Proposition 1.1 in [H96a]. Although the results in this paper proven by other methods covers the principal assertion of [H96a], we would like to give the reader a description of valid assertions of [H96a] and would like to correct false statements there.

We correct the statements of Proposition 1.1, Theorems 3.2 and 3.3 of [H96a] and give a corrected proof of them along the line employed in [H96a]. We use the notation introduced in [H96a]. Here is the corrected statement of Proposition 1.1 in [H96a]:

PROPOSITION 8.1: *Suppose the surjectivity of θ and μ. Then we have the following canonical exact sequence of H-modules:*

$$\mathrm{Tor}_1^H(B,\mathrm{Ker}(\mu)) \to C_1(\theta;T)\otimes_T B \to C_1(\lambda;B) \to C_1(\mu;B) \to 0.$$

In [H96a], the first term of the above exact sequence is written as $\mathrm{Tor}_1^T(B,\mathrm{Ker}(\mu))$. The proof given in [H96a] gives the correct result without any change. The mis-statement of this result affects the assertions made at several other places of [H96a]. Here is the corrected statement of Theorem 3.2 of [H96a]:

THEOREM 8.2: *Suppose $(\mathrm{AI}_{\mathbb{Q}})$, the conditions of $\mathcal{D}$ for $\overline{\rho}$ and that $\mathbb{I}$ is a torsion-free Λ_0-module of finite type giving the normalization of an irreducible component of $\mathrm{Spec}(R_{\mathbb{Q}})$. Let $\mathrm{Sel}^*_{\mathcal{L}}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$ be the Pontryagin dual module of the*

Selmer group $\mathrm{Sel}_{\mathcal{L}}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}$. *We have the following two exact sequences of* $\mathbb{I}$*-modules:*

$$\mathbb{I}\otimes_{\mathbb{Z}_p}\Gamma_j \xrightarrow{\varepsilon_j} \frac{\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}}{(\gamma^{p^j}-1)\,\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}}} \to C_1(\lambda_j;\mathbb{I})\to 0,$$
$$C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I} \xrightarrow{\iota_\infty} \mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi)\otimes\nu^{-1})_{/\mathbb{Q}} \to \mathbb{I}\to 0.$$

Moreover suppose that $R_{\mathbb{Q}}$ *is reduced and either that* $R_{\mathbb{Q}}$ *is a* Λ*-module of finite type or that* $\mathrm{Spec}(\mathbb{I}_0)$ *is an irreducible component of* $\mathrm{Spec}(R_{\mathbb{Q}})$. *Then* ε_j *is injective.*

In the original version in [H96a], it is claimed that $\mathrm{Ker}(\iota_\infty)$ is a pseudo-null $\mathbb{I}[[\Gamma]]$-module, which does not immediately follow from the method employed in [H96a]. Thus the analysis of $\mathrm{Ker}(\iota_j)$ given from the line 10 from the bottom of page 105 of [H96a] to the line 16 from the bottom of page 106 does not stand as it is. Removing this part from the proof, we get the corrected assertion.

We also need to correct the assertion of Theorem 3.3 in [H96a]. Here is the corrected one:

THEOREM 8.3: *Suppose* ($\mathrm{AI}_{\mathbb{Q}}$), (Ind), *that* $\mathbb{I}$ *is a torsion-free* Λ*-module of finite type giving the normalization of an irreducible component of* $\mathrm{Spec}(R_{\mathbb{Q}})$ *and that* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ *is a torsion* $\mathbb{I}$*-module. Then we have*

(i) $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *is a torsion* $\mathbb{I}[[\Gamma]]$*-module of finite type;*

(ii) *There is a pseudo-isomorphism of* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *into* $M\times\mathbb{I}$ *for a torsion* $\mathbb{I}[[\Gamma]]$*-module* $M = C_1(\lambda'_\infty;\mathbb{I})$ *such that* $M/(\gamma-1)M$ *is a torsion* $\mathbb{I}$*-module;*

(iii) *If* $\mathrm{Sel}^*_{\mathcal{M}'}(\mathrm{Ad}(\varphi))_{/\mathbb{Q}}$ *is a pseudo-null* $\mathbb{I}$*-module and* $\Lambda' = \mathbb{I}$, *then* $\mathrm{Sel}^*_{\mathcal{M}}(\mathrm{Ad}(\varphi))\otimes\nu^{-1})_{/\mathbb{Q}}$ *is pseudo-isomorphic to* $\mathbb{I}$, *on which* Γ *acts trivially;*

(iv) *If* $\mathbb{I}_0$ *is formally smooth over* $\mathfrak{O}$, *then we have the following exact sequence of* $\mathbb{I}[[\Gamma]]$*-modules:*

$$0\to C_1(\pi_\infty;\mathbb{I})\to C_1(\lambda'_\infty;\mathbb{I})\to\widehat{\Omega}_{\mathbb{I}/\Lambda'}\to 0,$$

where $\widehat{\Omega}_{\mathbb{I}/\Lambda'}$ *is the module of continuous* 1*-differentials or equivalently is the* $m_{\mathbb{I}}$*-adic completion of* $\Omega_{\mathbb{I}/\Lambda'}$ *(which is a torsion* $\mathbb{I}$*-module of finite type by* (Ind)*).*

Originally M is claimed to be pseudo-isomorphic to $C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}$ in the assertion (ii). This is true if $\mathbb{I}_0$ is formally smooth over $\mathfrak{O}$, and in this case, M is isomorphic to $C_1(\pi_\infty;\mathbb{I}_0)\otimes_{\mathbb{I}_0}\mathbb{I}$; otherwise, the proof given there does not immediately show the pseudo-isomorphism. The two arguments given after Theorem 3.3 in [H96a] proving the control of $C_1(\lambda'_j;\mathbb{I})$ and the $\mathbb{I}[[\Gamma]]$-torsion-ness of $C_1(\lambda'_\infty;\mathbb{I})$

are correct. However, the argument from the line 19 of page 109 of [H96a] to the line 3 from the bottom of the same page, relating $C_1(\lambda'_\infty;\mathbb{I})$ and $C_1(\pi_\infty;\mathbb{I})\otimes_{\mathbb{I}_0}\mathbb{I}$ up to $\mathbb{I}[[\Gamma]]$-pseudo null modules, is incorrect. The result holds when $\mathbb{I}$ is formally smooth as later proved in [H96a] pages 112–113. To recover the result (ii), we need to show that

$$0\to Y\to C_1(\lambda_\infty;\mathbb{I})\to C_1(\lambda'_\infty;\mathbb{I})\to 0$$

is exact for an $\mathbb{I}[[\Gamma]]$-torsion module Y. This can be done as follows: Note that

$$C_1(\lambda'_j;\mathbb{I})\cong\widehat{\Omega}_{R_j/\Lambda'_j}\otimes_{R_j}\mathbb{I}\quad\text{and}\quad C_1(\lambda_j;\mathbb{I})\cong\widehat{\Omega}_{R_j/\Lambda_j}\otimes_{R_j}\mathbb{I}.$$

We have by definition $\Lambda_\infty\cong\mathfrak{O}$ and $\Lambda'_\infty\cong\mathfrak{O}[[X]]$ by (Ind) of [H96a] page 107. Then the exact sequence:

$$\mathbb{I}\cong\widehat{\Omega}_{\mathfrak{O}[[X]]/\mathfrak{O}}\otimes_{\mathfrak{O}[[X]]}\mathbb{I}\to\widehat{\Omega}_{R_\infty/\Lambda_\infty}\otimes_{R_\infty}\mathbb{I}\to\widehat{\Omega}_{R_\infty/\Lambda'_\infty}\otimes_{R_\infty}\mathbb{I}\to 0$$

shows that Y is the image of $\mathbb{I}$, which is a torsion $\mathbb{I}[[\Gamma]]$-module. In this way, we can recover the assertion of Theorem 3.3 in [H96a] as stated above.

Here we list minor mistakes and misprints in [H96a]:

page 92 (Ext2):	$\mathrm{Tor}_1^{B'}$	should read	$\mathrm{Tor}_1^{T'}$ for $T'=T\otimes_A B$
page 100 line 6:	$T=R_E\otimes_{\Lambda'_E}\mathbb{I}$	should read	$T=R_F\otimes_{\Lambda'_F}\mathbb{I}$
page 105 (Ext5-6):	$\mathrm{Tor}_1^{T_j}$	should read	$\mathrm{Tor}_1^{T_k}$
page 116 lines 13:	$c(h\tau)c(\tau)$	should read	$c(h\tau)=\pi(h)c(\tau)$
page 127 Proposition A.2.3:			Remove $R_G^{\chi,ord}$ from the statement.

References

[BDGP] K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert, *Une preuve de la conjecture de Mahler-Manin,* Inventiones Mathematicae **124** (1996), 1–9.

[BR] D. Blasius and J. D. Rogawski, *Motives for Hilbert modular forms,* Inventiones Mathematicae **114** (1993), 55–87.

[Cr] H. Carayol, *Formes modulaires et représentations galoisiennes à valeurs dans un anneau local compact,* Contemporary Mathematics **165** (1994), 213–237.

[DM] P. Deligne and J. Milne, *Tannakian categories,* Lecture Notes in Mathematics **900**, Springer-Verlag, Berlin, 1982, pp. 101–228.

[DT] F. Diamond and R. Taylor, *Lifting modular mod l representations,* Duke Mathematical Journal **74** (1994), 253–269.

[DHI] K. Doi, H. Hida and H. Ishii, *Discriminant of Hecke fields and the twisted adjoint L-values for* GL(2), Inventiones Mathematicae **134** (1998), 547–577.

[Fu] K. Fujiwara, *Deformation rings and Hecke algebras in totally real case*, preprint, 1996.

[Gr] R. Greenberg, *Iwasawa theory and p-adic deformation of motives*, Proceedings of Symposia in Pure Mathematics **55**, Part 2 (1994), 193–223.

[Gr1] R. Greenberg, *Arithmetic Theory of Elliptic Curves*, preprint, to appear in Lecture Notes in Mathematics, Springer.

[GS] R. Greenberg and G. Stevens, *On the conjecture of Mazur, Tate, and Teitelbaum*, Contemporary Mathematics **165** (1994), 183–211.

[HaT] M. Harris and R. Taylor, *Deformation of automorphic Galois representations*, preprint, 1998.

[H88] H. Hida, *On p-adic Hecke algebras for GL_2 over totally real fields*, Annals of Mathematics **128** (1988), 295–384.

[H89a] H. Hida, *On nearly ordinary Hecke algebras for* GL(2) *over totally real fields*, Advanced Studies in Pure Mathematics **17** (1989), 139–169.

[H89b] H. Hida, *Nearly ordinary Hecke algebras and Galois representations of several variables*, Proceedings of JAMI Inaugural Conference, Supplement to American Journal of Mathematics (1989), 115–134.

[H90] H. Hida, *p-adic L-functions for base change lifts of* GL_2 *to* GL_3, in Proceedings of Conference on "Automorphic forms, Shimura varieties, and L-functions", Perspectives in Mathematics **11** (1990), 93–142.

[H93] H. Hida, *Elementary Theory of L-functions and Eisenstein Series*, London Mathematical Society Student Texts **26**, Cambridge University Press, 1993.

[H96a] H. Hida, *On Selmer groups of adjoint modular Galois representations*, in *Number Theory, Paris*, London Mathematical Society Lecture Notes Series **235** (1996), 89–132.

[H96b] H. Hida, *On the search of genuine p-adic modular L-functions for* GL(n), Mémoires de la Société Mathématique de France **67** (1996).

[H97a] H. Hida, *Hecke algebras, Selmer groups and base change*, preprint, 1997.

[H97b] H. Hida, *Modular adjoint Selmer groups of several variables over totally real fields*, preprint, 1997.

[H99a] H. Hida, *Non-critical values of adjoint L-functions for* SL(2), Proceedings of Symposia in Pure Mathematics **66** (1999), Part 1, 123–175.

[H99b] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge University Press, 1999.

[H99c] H. Hida, *Control theorems for coherent sheaves on Shimura vrieties of PEL-type*, preprint, 1999.

[HM] H. Hida and Y. Maeda, *Non-abelian base change for totally real fields*, Special issue of Pacific Journal of Mathematics in memory of Olga Taussky Todd (1997), 189–217.

[HT] H. Hida and J. Tilouine, *On the anticyclotomic main conjecture for CM fields*, Inventiones Mathematicae **117** (1994), 89–147.

[HTU] H. Hida, J. Tilouine and E. Urban, *Adjoint modular Galois representations and their Selmer groups*, Proceedings of the National Academy of Sciences of the United States of America **94** (1997), 11121–11124.

[L] R. P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies **96**, Princeton University Press, 1980.

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1989.

[MT] B. Mazur and J. Tilouine, *Représentations Galoisiennes différentielles de Kähler et "conjectures principales"*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **71** (1990), 65–103.

[MW] B. Mazur and A. Wiles, *On p-adic analytic families of Galois representations*, Compositio Mathematica **59** (1986), 231–264.

[O] T. Ochiai, *Control theorem for Selmer groups of p-adic representations*, Master Thesis, University of Tokyo, 1998.

[T] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Inventiones Mathematicae **98** (1989), 265–280.

[TW] R. Taylor and A. Wiles, *Ring theoretic properties of certain Hecke modules*, Annals of Mathematics **142** (1995), 553–572.

[Ti] J. Tilouine, *Un sous-groupe p-divisible de la jacobienne de $X_1(Np^r)$ comme module sur l'algèbre de Hecke*, Bulletin de la Société Mathématique de France **115** (1987), 329–360.

[Ti1] J. Tilouine, *Deformation of Galois representations and Hecke algebras*, Publications of the Mehta Research Institute, Narosa Publications, Delhi, 1996.

[U] E. Urban, *Selmer groups and the Eisenstein-Klingen ideal*, preprint, 1997.

[W] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Annals of Mathematics **142** (1995), 443–551.